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Existence of a Stochastic Model of an Age-Structured Diphtheria Infection

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Abstract

In this research, a stochastic model is developed and analysed for the age structured deterministic model of diphtheria infection. The model includes specific epidemiological classes such as S_1 Susceptible infant at time t (0-1years), S_2 , Susceptible school children population at time t, V, Vaccinated population at time t, E, Exposed population at time t , I_1, Asymptomatic infected population at time t, I_2., symptomatic infected population at time t, I_D., Detected infectious humans at time t(Asymptomatic and symptomatic) population through testing, R., Recovered population at time t . The work investigates the transformation of age-structured deterministic model of diphtheria infection into an age-structured stochastic model of diphtheria infection , by applying the well established procedure proposed by [1] and also investigates the existence and uniqueness of solution of the transformed stochastic model. It is established that developed stochastic model of diphtheria infection dynamics satisfies the condition for the existence and uniqueness of solution. The numerical simulation of the work; shows that, increased vaccination rate will lead to Diphtheria infection extinction while low vaccination rate will produce an epidemic, demonstrates how randomness can cause the outbreak of Diphtheria die out and indicates the effect of random noise in detecting the disease hence suppressing the initial infection wave.

Key words and phrases: Vaccination, transition probability, Diphtheria, stochastic process, probability distribution.

Introduction

The use of mathematical entities such as equations, variables, functions, symbols, etc provides a fundamental description of physical reality, and enhances a better understanding of underlying phenomena of the system under consideration [2], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13], [14], [15], [16].

Diphtheria is a bacterial infectious disease that scientists need to understand its transmission dynamics, with the aim of advancing it preventive and control strategies. Diphtheria infection can lead to severe complications such as respiratory failure, heart problems and even deaths if it is not detected early. Several researches have been carried out in recent time on the mathematical modelling of the transmission dynamic of diphtheria infection;

[13] developed an aged structured deterministic mathematical model of diphtheria infection. It was established through theorems and proofs that the model is epidemiologically meaningful, and that all its state variables are positive (non-negative) at time $t > 0$ in the domain $\wp$, and that the domain $\wp$ is indeed bounded. Using the next generation matrix, the reproduction ratio of the system was determined. Using dynamical system theory, it was established that the system is locally stable. A matrix theoretic method was used in the construction of an appropriate Lyapunov function for the global stability analysis of the formulated model, and also established that the system is globally asymptotically stable if $R_b \leq 1$ and unstable otherwise.

[14], used Optimal control theory to study the impact of personal protection, vaccination and treatment as effective control measures against diphtheria infection and established necessary conditions for the optimal control of the system. The application of time-dependent controls that can reduce the total number of infected (symptomatic and asymptomatic) individuals in the population were also established. To determine the most influential parameters of the model that greatly influence the prevalence of diphtheria, sensitivity analysis was carried out on R_b using the normalised forward sensitivity index of R_b that depends differentially on basic reproduction ration parameters. From the sensitivity analysis is was discovered that, the most influential parameters of the model that greatly influence the prevalence of diphtheria were effective rate of transmission, rate of progression from the latent state to the infected state, fraction of new infection that are both symptomatic and asymptomatic, detection rate, natural and infection death rate. The numerical simulations showed that there is a significant effect of vaccination and isolation for susceptible individuals. The number of susceptible infants reduces and drops to zero when each of the controls was applied with time. The use of treatment / drug inhibition control could significantly reduce the number of both asymptomatic and symptomatic infected individuals

The mentioned works and numerous studies such as [18], [19], [20], [21], [22], [23] are mostly deterministic, and assume that all input variables are deterministic functions of time, ignoring completely the randomness of these variables. Because the fundamental biological processes involved most real cases are stochastic, ignoring the inherent randomness in these processes may not be ideal. In this work, we seek to extend the work [15], [16], [13], [20] to capture the randomness inherent in the biological process of diphtheria infection, by converting [15], [16], [13], [20] to a stochastic model and establishing the condition for existence and uniqueness of solution of the developed stochastic model. This extension allows SDEs to model phenomena that cannot be accurately captured by deterministic equations alone.

A stochastic differential equation (SDE) model describes dynamical systems influenced by random noise, extending deterministic differential equations by adding a stochastic process term, often Brownian motion, allowing it to model systems influenced by uncertainty and random fluctuations. The core components of a Stochastic Differential Equations include the drift term,

the diffusion term, and the Wiener process. The **drift term** represents the deterministic part of the equation, often linked to the average rate of change in the system. The **diffusion term** captures the random fluctuations, scaling the impact of the stochastic component. The **Wiener process**, or Brownian motion, is a mathematical representation of random movement, providing the stochastic input that drives the system's behavior.

2.0 MODEL FORMULATION

2.1 ASSUMPTIONS

1. The control of diphtheria is based on primary prevention of disease by ensuring high population immunity of the infant ((0-1 year) and school children by vaccination
2. Isolation of detected cases (that is confirmed cases are not allowed to interact with the population freely.
3. Epidemiological surveillance ensuring early detection through contact tracing is carried out
4. Secondary prevention spread by the rapid investigation of close contacts to ensure prompt treatment of the infected
5. The total population of human at time t under consideration denoted by N_h is split into mutually exclusive sub-population of S_1 Susceptibleinfantat time t (0-1years), S_2 , Susceptible school children populationat time t, V, Vaccination populationat time t, E, Exposed populationat time t , I_1, Asymptomatic infection populationat time t, I_2., symptomatic infection populationat time t, R., Recovered populationat time t. I_D., Detected infectious humansat time t(Asymptomatic and symptomatic) population through testing,

$$H_h = S_1 + S_2 + V + E + I_1 + I_2 + R + I_D$$

2.2 STATE VARIABLES

N_h−The total population of human at time t under consideration
S_1− Susceptible infant at time t (0-1years),
S_2− Susceptible school children population at time t,
V−Vaccination population at time t,
E− Exposed population at time ,
I_1 − Asymptomatic infection population at time t,
I_2 − Symptomatic infection population at time t,
I_D− Detected infectious humans at time t(Asymptomatic and symptomatic) population through testing,
R−Recovered population at time t

2.3 PARAMETERS

$$\sigma_1 - progress\ rate\ from\ exposed\ to\ I_1,\ Asymptomatic\ infected\ population$$

$$\sigma_2 - progress\ rate\ efrom\ exposed\ to\ I_2,\ symptomatic\ infection\ population$$

$$\delta - fraction\ of\ new\ infection\ that\ are\ I_1,\ Asymptomatic$$

$$1-\delta - fraction\ of\ new\ infection\ that\ are\ I_2,\ symptomatic$$

$$\beta_1 - effective\ transmission\ rate\ from\ S_1$$

$$\beta_2 - effective\ transmission\ rate\ from\ S_2$$

$$\tau_1 - vaccination\ coverage\ \ for\ \ S_1$$

$$\tau_2 - vaccination\ coverage\ \ for\ \ S_2$$

$$\varepsilon - vaccine\ efficacy$$

$$\rho - maturity\ rate\ from\ \ S_1\ to\ S_2$$

$$b - per\ capita\ birth\ rate\ of\ humans$$

$$\varepsilon\tau_1 - immunization\ rate\ in\ \ S_1$$

$$\varepsilon\tau_2 - immunization\ rate\ in\ \ S_2$$

$$\theta_1 - detection\ rate\ (via\ contact\ tracing\ )\ \ for\ \ S_1$$

$$\theta_2 - detection\ rate\ (via\ contact\ tracing\ )\ \ for\ \ S_2$$

$$\eta - death\ due\ to\ infection$$

$$\alpha - natural\ death\ rate$$

$$\gamma_1 - the\ recovery\ rate\ of\ \ I_1$$

$$\gamma_2 - the\ recovery\ rate\ of\ \ I_2$$

$$\gamma_3 - the\ recovery\ rate\ of\ \ I_D$$

$$\lambda_1(t) = \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}$$

$$\lambda_2(t) = \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}$$

2.4 MODEL EQUATIONS

Using the above described state variables and parameters, the model of the diphtheria infection transmission dynamics results in the following system of deterministicnon-linear first order differential equations

$$\frac{dS_1}{dt} = bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1 \tag{2.1}$$

$$\frac{dS_2}{dt} = \rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2 \tag{2.2}$$

$$\frac{dV}{dt} = \varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \alpha V \tag{2.3}$$

$$\frac{dE}{dt} = \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \sigma_1\delta E - \sigma_2(1-\delta)E - \alpha E \tag{2.4}$$

$$\frac{dI_1}{dt} = \sigma_1\delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1 \tag{2.5}$$

$$\frac{dI_2}{dt} = \sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2 \tag{2.6}$$

$$\frac{dI_D}{dt} = \theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D \tag{2.7}$$

$$\frac{dR}{dt} = \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R \tag{2.8}$$

The transition probabilities are shown in table 2.1

Tab 2.1: THE TRANSITION PROBABILITIES

CHANGE	PROBABILITY	EVENT
$[1\,0\,0\,0\,0\,0\,0\,0]^T$	$P_1 = bN_h\Delta t$	Recruitment of susceptible humans
$[-1\,0\,0\,1\,0\,0\,0\,0]^T$	$P_2 = \frac{\beta_1\sigma_2 I_2 S_1}{N_h - I_D}\Delta t$	Infant becomes symptomatically infected
$[-1\,0\,0\,1\,0\,0\,0\,0]^T$	$P_3 = \frac{\beta_1\sigma_1 I_1 S_1}{N_h - I_D}\Delta t$	Infant becomes asymptomatically infected
$[0-1\,0\,1\,0\,0\,0\,0]^T$	$P_4 = \frac{\beta_2\sigma_2 I_2 S_2}{N_h - I_D}\Delta t$	School children become symptomatically infected
$[0-1\,0\,1\,0\,0\,0\,0]^T$	$P_5 = \frac{\beta_2\sigma_1 I_1 S_2}{N_h - I_D}\Delta t$	School children become asymptomatically infected
$[-1\,0\,1\,0\,0\,0\,0\,0]^T$	$P_6 = \varepsilon\tau_1 S_1\Delta t$	The immunization rate of infants
$[-1\,1\;\;0\;\;0\,0\,0\,0\,0]^T$	$P_7 = \rho S_1\Delta t$	*maturity rate from* S_1 *to* S_2
$[0-1\,1\,0\,0\,0\,0\,0]^T$	$P_8 = \varepsilon\tau_2 S_2\Delta t$	The immunization rate of school children
$[0\;-1\;\;0\,0\,0\,0\,0\,0]^T$	$P_9 = \alpha S_2\Delta t$	Natural death of school children
$[-1\,0\;\;0\,0\,0\,0\,0\,0]^T$	$P_{10} = \alpha S_1\Delta t$	Natural death of infants
$[0\;\;0\;-1\,0\,0\,0\,0\,0]^T$	$P_{11} = \alpha V\Delta t$	Natural death of the vaccinated
$[0\;\;0\,0\;-1\,1\,0\,0\,0]^T$	$P_{12} = \sigma_1\delta E\Delta t$	Exposed Asymptomaticallyinfected
$[0\;\;0\,0\;-1\,0\,1\,0\,0]^T$	$P_{13} = \sigma_2(1-\delta)E\Delta t$	Exposed symptomatically infected

CHANGE	PROBABILITY	EVENT
$[0\ 0\ 0\ 0\ -1\ 0\ 0\ 1]^T$	$P_{14} = \gamma_1 I_1 \Delta t$	Asymptomatic infected recovers
$[0\ 0\ 0\ 0\ -1\ 0\ 1\ 0]^T$	$P_{15} = \theta_1 I_1 \Delta t$	Detection of asymptomatic infection
$[0\ 0\ 0\ 0\ -1\ 0\ 0\ 0]^T$	$P_{16} = \alpha I_1 \Delta t$	Natural death of the asymptomatic
$[0\ 0\ 0\ 0\ -1\ 0\ 0\ 0]^T$	$P_{17} = \eta I_1 \Delta t$	The asymptomatically infected dies due to infection
$[0\ 0\ 0\ 0\ 0\ -1\ 0\ 0]^T$	$P_{18} = \alpha I_2 \Delta t$	Natural death of the symptomatically infected
$[0\ 0\ 0\ 0\ 0\ -1\ 0\ 0]^T$	$P_{19} = \eta I_2 \Delta t$	The symptomatically infected dies due to infectio
$[0\ 0\ 0\ 0\ 0\ -1\ 0\ 1]^T$	$P_{20} = \gamma_2 I_2 \Delta t$	Symptomatic infected recovers
$[0\ 0\ 0\ 0\ 0\ -1\ 1\ 0]^T$	$P_{21} = \theta_2 I_2 \Delta t$	Detection of symptomatic infection
$[0\ 0\ 0\ 0\ 0\ 0\ -1\ 1]^T$	$P_{22} = \gamma_3 I_D \Delta t$	Detected infected recovers
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ -1]^T$	$P_{23} = \alpha R \Delta t$	Natural death of the removed
$[0\ 0\ 0\ 0\ 0\ 0\ -1\ 0]^T$	$P_{24} = \eta I_D \quad \Delta t$	Death of the detected due to infection
$[0\ 0\ 0\ 0\ 0\ 0\ -1\ 0]^T$	$P_{25} = \alpha I_D \Delta t$	Natural death of the detected
$[0\ 0\ 0\ -1\ 0\ 0\ 0\ 0]^T$	$P_{26} = \alpha E \Delta t$	Natural death of the exposed

2.5 The Stochastic Model Equations (SMEs)

Using the second modelling procedure developed by [1] on the construction of stochastic differential equation from the deterministic differential equation, hence the stochastic model equation are given as

$$\begin{aligned} d\overline{X}(t) &= \overline{f}(t,\overline{X}(t))\,dt + G(t,\overline{X}(t))\,dW_t, \\ \overline{X}(0) &= \big(S_1(0), S_2(0), V(0), E(0), I_1(0), I_2(0), I_D(0), R(0)\big)^T. \end{aligned} \tag{2.5.1}$$

Let $\lambda_j \in \mathbb{R}^8$ denote the change vector for event j, and let $r_j(X) = P_j(X)/\Delta t$ be its transition rate. With independent standard Brownian motions $W_1, \ldots, W_{26}$, the event-based stochastic system is

$$dX_t = \sum_{j=1}^{26} \lambda_j r_j(X_t)\,dt + \sum_{j=1}^{26} \lambda_j \sqrt{r_j(X_t)}\,dW_j(t), \tag{2.5.2}$$

where $W_t = (W_1(t), \ldots, W_{26}(t))^T$. Thus, the columns of G are $\lambda_j \sqrt{r_j(X)}$, in accordance with the transition table.

The drift vector is defined as

$$\overrightarrow{f} = \sum_{j=1}^{26} r_j(X)\,\overrightarrow{\lambda}_j \tag{2.5.3}$$

Here, $r_j(X) = P_j(X)/\Delta t$ is the event rate and $\overrightarrow{\lambda}_j$ is its state-change vector (Table 2.1).

The drift vector is given as

$$
\begin{aligned}
f_1 &= bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1, \\
f_2 &= \rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2, \\
f_3 &= \varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \alpha V, \\
f_4 &= \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 \\
&\quad - \sigma_1\delta E - \sigma_2(1-\delta)E - \alpha E, \\
f_5 &= \sigma_1\delta E - (\gamma_1 + \theta_1 + \alpha + \eta) I_1, \\
f_6 &= \sigma_2(1-\delta)E - (\gamma_2 + \theta_2 + \alpha + \eta) I_2, \\
f_7 &= \theta_1 I_1 + \theta_2 I_2 - (\gamma_3 + \alpha + \eta) I_D, \\
f_8 &= \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R.
\end{aligned}
\tag{2.5.4}
$$

The diffusion matrix is constructed from the event rates and state-change vectors. Its column for event j is $\lambda_j \sqrt{r_j(X)}$. For readability, the 8×26 matrix is displayed in two column blocks:

Columns 1–13 of the diffusion matrix

$$\begin{pmatrix}
\sqrt{g_{11}} & -\sqrt{g_{12}} & -\sqrt{g_{13}} & 0 & 0 & -\sqrt{g_{16}} & -\sqrt{g_{17}} & 0 & 0 & -\sqrt{g_{110}} & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{g_{24}} & -\sqrt{g_{25}} & 0 & \sqrt{g_{27}} & -\sqrt{g_{28}} & -\sqrt{g_{29}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{g_{36}} & 0 & \sqrt{g_{38}} & 0 & 0 & -\sqrt{g_{311}} & 0 & 0 \\
0 & \sqrt{g_{42}} & \sqrt{g_{43}} & \sqrt{g_{44}} & \sqrt{g_{45}} & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{g_{412}} & -\sqrt{g_{413}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{g_{512}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{g_{613}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Columns 14–26 of the diffusion matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{g_{426}} \\
-\sqrt{g_{514}} & -\sqrt{g_{515}} & -\sqrt{g_{516}} & -\sqrt{g_{517}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{g_{618}} & -\sqrt{g_{619}} & -\sqrt{g_{620}} & -\sqrt{g_{621}} & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{g_{715}} & 0 & 0 & 0 & 0 & 0 & \sqrt{g_{721}} & -\sqrt{g_{722}} & 0 & -\sqrt{g_{724}} & -\sqrt{g_{725}} & 0 \\
\sqrt{g_{814}} & 0 & 0 & 0 & 0 & 0 & \sqrt{g_{820}} & 0 & \sqrt{g_{822}} & -\sqrt{g_{823}} & 0 & 0 & 0
\end{pmatrix}$$

Here, g_{ij} denotes the rate of event j contributing to state equation i. The nonzero rates are

$$
\begin{aligned}
&g_{11} = bN_h, && g_{12} = \frac{\beta_1\sigma_2 I_2 S_1}{N_h - I_D}, && g_{13} = \frac{\beta_1\sigma_1 I_1 S_1}{N_h - I_D},\\
&g_{16} = \varepsilon\tau_1 S_1, && g_{17} = \rho S_1, && g_{110} = \alpha S_1,\\
&g_{24} = \frac{\beta_2\sigma_2 I_2 S_2}{N_h - I_D}, && g_{25} = \frac{\beta_2\sigma_1 I_1 S_2}{N_h - I_D}, && g_{27} = \rho S_1,\\
&g_{28} = \varepsilon\tau_2 S_2, && g_{29} = \alpha S_2,\\
&g_{36} = \varepsilon\tau_1 S_1, && g_{38} = \varepsilon\tau_2 S_2, && g_{311} = \alpha V,\\
&g_{42} = \frac{\beta_1\sigma_2 I_2 S_1}{N_h - I_D}, && g_{43} = \frac{\beta_1\sigma_1 I_1 S_1}{N_h - I_D},\\
&g_{44} = \frac{\beta_2\sigma_2 I_2 S_2}{N_h - I_D}, && g_{45} = \frac{\beta_2\sigma_1 I_1 S_2}{N_h - I_D},\\
&g_{412} = \sigma_1\delta E, && g_{413} = \sigma_2(1-\delta)E,\\
&g_{512} = \sigma_1\delta E, && g_{514} = \gamma_1 I_1, && g_{515} = \theta_1 I_1,\\
&g_{516} = \alpha I_1, && g_{517} = \eta I_1,\\
&g_{613} = \sigma_2(1-\delta)E, && g_{618} = \alpha I_2, && g_{619} = \eta I_2,\\
&g_{620} = \gamma_2 I_2, && g_{621} = \theta_2 I_2,\\
&g_{715} = \theta_1 I_1, && g_{721} = \theta_2 I_2, && g_{722} = \gamma_3 I_D,\\
&g_{724} = \eta I_D, && g_{725} = \alpha I_D, && g_{426} = \alpha E,\\
&g_{814} = \gamma_1 I_1, && g_{820} = \gamma_2 I_2, && g_{822} = \gamma_3 I_D, && g_{823} = \alpha R.
\end{aligned}
$$

The probability distribution $(t,)$ of the solution that solves the above stochastic differential equation satisfies the forward kolmogorov equation given below:

$$
\begin{aligned}
\frac{\partial p(t,x)}{\partial t} &= -\sum_{i=1}^{8}\frac{\partial[p(t,x)f_i(t,x)]}{\partial x_i}\\
&\quad + \frac{1}{2}\sum_{i=1}^{8}\sum_{k=1}^{8}\sum_{\ell=1}^{26}\frac{\partial^2[p(t,x)G_{i\ell}(t,x)G_{k\ell}(t,x)]}{\partial x_i\partial x_k}.
\end{aligned}
\tag{2.5.5}
$$

2.6 Existence and Uniqueness theorem for SDEs ([17])

Assume that the coefficients in the following system of stochastic differential equations

$$
\mathrm{d}X_t^i = a_i(t,X_t)\,\mathrm{d}t + \sum_{j=1}^{m} b_{ij}(t,X_t)\,\mathrm{d}W_t^j. \tag{2.6.1}
$$

Where $X_t^i = (X_t^1,\ X_t^2,\ X_t^3,\ .\ .\ .,\ X_t^n)^T$, $W_t^j = (W_t^1,\ W_t^2,\ W_t^3,\ .\ .\ .,\ W_t^m)^T$

$$
a_i\,(t,\ X_t) -\ \textit{is an } n-\textit{dimensional vector with entries } a_i(t,\ x)
$$

$$
b_{ij}\,(t,\ X_t) -\ \textit{is an } n\times m \textit{ matrix with entries } b_{ij}(t,\ x)
$$

Satisfying the following Lipschitz condition and growth conditions in (2.6.2) for some constant $q\ <\infty$ and for *all* $t\in\Re$ *and* $x,y\ \varepsilon\ \Re^n$

$$
\begin{aligned}
&\|a(t,x)-a(t,y)\| \le q\|x-y\|,\\
&\|b(t,x)-b(t,y)\| \le q\|x-y\|,\\
&\|a(t,x)\| \le q(1+\|x\|), \qquad \|b(t,x)\| \le q(1+\|x\|).
\end{aligned}
\tag{2.6.2}
$$

$$\|b\| = \sqrt{\sum_{i}^{n}\sum_{j}^{m} b_{ij}(x)^2}$$

$$\|a\| = \sqrt{\sum_{j}^{n} a_i(x)^2}$$

Then for each $x_0 \in \Re^n$ there is a unique solution to the system of stochastic differential equations (2.15) such that $X_0 = x_0$

Now consider equations (2.5.1) and (2.5.2) with the following:

$$X(0) = [S_1(0),\ S_2(0),\ V(0),\ E(0),\ I_1(0),\ I_2(0),\ I_D(0),\ R(0)]$$

$$f_1 = bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1$$

$$f_2 = \rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2$$

$$f_3 = \varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \alpha V$$

$$f_4 = \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \sigma_1\delta E - \sigma_2(1-\delta)E - \alpha E.$$

$$f_5 = \sigma_1\delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1$$

$$f_6 = \sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2$$

$$f_7 = \theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D$$

$$f_8 = \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R$$

On a compact domain $D \subset \{x_i > 0,\ N_h - I_D \geq c_0\}$, where $c_0 > 0$, the drift derivatives are bounded; hence there is a constant K such that

$$\left|\frac{\partial f_1}{\partial S_1}\right| = \left|\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} + \varepsilon\tau_1 + \rho + \alpha\right| \leq K,$$
$$\left|\frac{\partial f_1}{\partial I_1}\right| = \left|\frac{\beta_1\sigma_1 S_1}{N_h - I_D}\right| \leq K, \quad \left|\frac{\partial f_1}{\partial I_2}\right| = \left|\frac{\beta_1\sigma_2 S_1}{N_h - I_D}\right| \leq K,$$
$$\left|\frac{\partial f_1}{\partial I_D}\right| = \left|\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)S_1}{(N_h - I_D)^2}\right| \leq K,$$
$$\frac{\partial f_1}{\partial S_2} = \frac{\partial f_1}{\partial V} = \frac{\partial f_1}{\partial E} = \frac{\partial f_1}{\partial R} = 0,$$
$$\left|\frac{\partial f_2}{\partial S_1}\right| = |\rho| \leq K,$$
$$\left|\frac{\partial f_2}{\partial S_2}\right| = \left|\frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} + \varepsilon\tau_2 + \alpha\right| \leq K,$$

$$\left|\frac{\partial f_2}{\partial I_1}\right| = \left|\frac{\beta_2\sigma_1 S_2}{N_h - I_D}\right| \leq K, \quad \left|\frac{\partial f_2}{\partial I_2}\right| = \left|\frac{\beta_2\sigma_2 S_2}{N_h - I_D}\right| \leq K,$$
$$\left|\frac{\partial f_2}{\partial I_D}\right| = \left|\frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)S_2}{(N_h - I_D)^2}\right| \leq K,$$
$$\frac{\partial f_2}{\partial V} = \frac{\partial f_2}{\partial E} = \frac{\partial f_2}{\partial R} = 0,$$
$$\left|\frac{\partial f_3}{\partial S_1}\right| = |\varepsilon\tau_1| \leq K, \quad \left|\frac{\partial f_3}{\partial S_2}\right| = |\varepsilon\tau_2| \leq K,$$
$$\left|\frac{\partial f_3}{\partial V}\right| = |\alpha| \leq K,$$
$$\frac{\partial f_3}{\partial E} = \frac{\partial f_3}{\partial R} = \frac{\partial f_3}{\partial I_D} = \frac{\partial f_3}{\partial I_1} = \frac{\partial f_3}{\partial I_2} = 0,$$
$$\left|\frac{\partial f_4}{\partial S_1}\right| = \left|\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}\right| \leq K,$$
$$\left|\frac{\partial f_4}{\partial S_2}\right| = \left|\frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}\right| \leq K,$$
$$\left|\frac{\partial f_4}{\partial E}\right| = |\sigma_1\delta + \sigma_2(1-\delta) + \alpha| \leq K,$$
$$\left|\frac{\partial f_4}{\partial I_D}\right| = \left|\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)S_1 + \beta_2(\sigma_1 I_1 + \sigma_2 I_2)S_2}{(N_h - I_D)^2}\right| \leq K,$$
$$\left|\frac{\partial f_4}{\partial I_1}\right| = \left|\frac{\beta_1\sigma_1 S_1 + \beta_2\sigma_1 S_2}{N_h - I_D}\right| \leq K,$$
$$\left|\frac{\partial f_4}{\partial I_2}\right| = \left|\frac{\beta_1\sigma_2 S_1 + \beta_2\sigma_2 S_2}{N_h - I_D}\right| \leq K,$$
$$\frac{\partial f_4}{\partial V} = \frac{\partial f_4}{\partial R} = 0,$$
$$\frac{\partial f_5}{\partial S_1} = \frac{\partial f_5}{\partial S_2} = \frac{\partial f_5}{\partial V} = 0, \quad \left|\frac{\partial f_5}{\partial E}\right| = |\sigma_1\delta| \leq K,$$
$$\left|\frac{\partial f_5}{\partial I_1}\right| = |\gamma_1 + \theta_1 + \alpha + \eta| \leq K, \quad \frac{\partial f_5}{\partial I_2} = \frac{\partial f_5}{\partial R} = \frac{\partial f_5}{\partial I_D} = 0,$$
$$\left|\frac{\partial f_6}{\partial E}\right| = |\sigma_2(1-\delta)| \leq K, \quad \left|\frac{\partial f_6}{\partial I_2}\right| = |\gamma_2 + \theta_2 + \alpha + \eta| \leq K,$$
$$\frac{\partial f_6}{\partial I_1} = \frac{\partial f_6}{\partial R} = \frac{\partial f_6}{\partial I_D} = \frac{\partial f_6}{\partial S_1} = \frac{\partial f_6}{\partial S_2} = \frac{\partial f_6}{\partial V} = 0,$$
$$\left|\frac{\partial f_7}{\partial I_1}\right| = |\theta_1| \leq K, \quad \left|\frac{\partial f_7}{\partial I_2}\right| = |\theta_2| \leq K,$$
$$\left|\frac{\partial f_7}{\partial I_D}\right| = |\gamma_3 + \alpha + \eta| \leq K,$$
$$\frac{\partial f_7}{\partial R} = \frac{\partial f_7}{\partial S_1} = \frac{\partial f_7}{\partial S_2} = \frac{\partial f_7}{\partial V} = \frac{\partial f_7}{\partial E} = 0,$$
$$\left|\frac{\partial f_8}{\partial R}\right| = |\alpha| \leq K, \quad \left|\frac{\partial f_8}{\partial I_1}\right| = |\gamma_1| \leq K,$$
$$\left|\frac{\partial f_8}{\partial I_2}\right| = |\gamma_2| \leq K, \quad \left|\frac{\partial f_8}{\partial I_D}\right| = |\gamma_3| \leq K,$$
$$\frac{\partial f_8}{\partial S_1} = \frac{\partial f_8}{\partial S_2} = \frac{\partial f_8}{\partial V} = \frac{\partial f_8}{\partial E} = 0.$$

Also, from (2.5.1) and (2.5.2),

$$\|f\| = \sqrt{\sum_{i=1}^{8} f_i(x)^2}.$$

The components of the drift vector can be written as

$$\begin{aligned}
f_1 &= bN_h - \left\{ \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} + \varepsilon\tau_1 + \rho + \alpha \right\} S_1, \\
f_2 &= \rho S_1 - \left\{ \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} + \varepsilon\tau_2 + \alpha \right\} S_2, \\
f_3 &= \varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \alpha V, \\
f_4 &= \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 \\
&\quad - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E, \\
f_5 &= \sigma_1\delta E - (\gamma_1 + \theta_1 + \alpha + \eta)I_1, \\
f_6 &= \sigma_2(1-\delta)E - (\gamma_2 + \theta_2 + \alpha + \eta)I_2, \\
f_7 &= \theta_1 I_1 + \theta_2 I_2 - (\gamma_3 + \alpha + \eta)I_D, \\
f_8 &= \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R.
\end{aligned}$$

And

$$\begin{aligned}
\|G\| &= \sqrt{\sum_{i=1}^{8}\sum_{j=1}^{26} G_{ij}^2(x)} \\
&= \sqrt{\begin{aligned}
&bN_h + 2\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + 2\varepsilon\tau_1 S_1 + 2\rho S_1 + \alpha S_1 \\
&\quad + 2\frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 + 2\varepsilon\tau_2 S_2 + \alpha S_2 + \alpha V \\
&\quad + 2\sigma_1\delta E + 2\sigma_2(1-\delta)E + \alpha E \\
&\quad + 2\gamma_1 I_1 + 2\theta_1 I_1 + (\alpha + \eta)I_1 + 2\gamma_2 I_2 \\
&\quad + 2\theta_2 I_2 + (\alpha + \eta)I_2 + 2\gamma_3 I_D + (\alpha + \eta)I_D + \alpha R
\end{aligned}}
\end{aligned}$$

On any compact interior region $D \subset \{x_i > 0,\ N_h - I_D \geq c_0\}$, with $c_0 > 0$, the drift and diffusion coefficients are locally Lipschitz and bounded. The standard theorem therefore gives a unique local solution up to the first exit from D. Since the diffusion entries contain square roots of event rates, this derivative calculation does not establish global Lipschitz continuity at the boundary of the nonnegative state space.

Numerical Simulation

Table 1 Model Parameters and values

Parameters	Values	Source
b	0.15448/dy	[24]
α	0.0000194	[24]
$\gamma_1, \gamma_2, \gamma_3$	0.5, 0.6, 0.7	Assumed
θ_1, θ_2	0.3, 0.4	Assumed
β_1, β_2	0.57	[25]
ε	0.6	Assumed
ρ	0.7	Assumed
σ_1, σ_2	0.143, 0.140	[24]
δ	0.2	Assumed
τ_1, τ_2	0.5, 0.4	[24]

Initial conditions for the model variables are

$$S_1(0) = 500, \quad S_2(0) = 500, \quad V(0) = 200, \quad E(0) = 700,$$
$$I_1(0) = 100, \quad I_2(0) = 200, \quad I_D(0) = 400, \quad R(0) = 400.$$

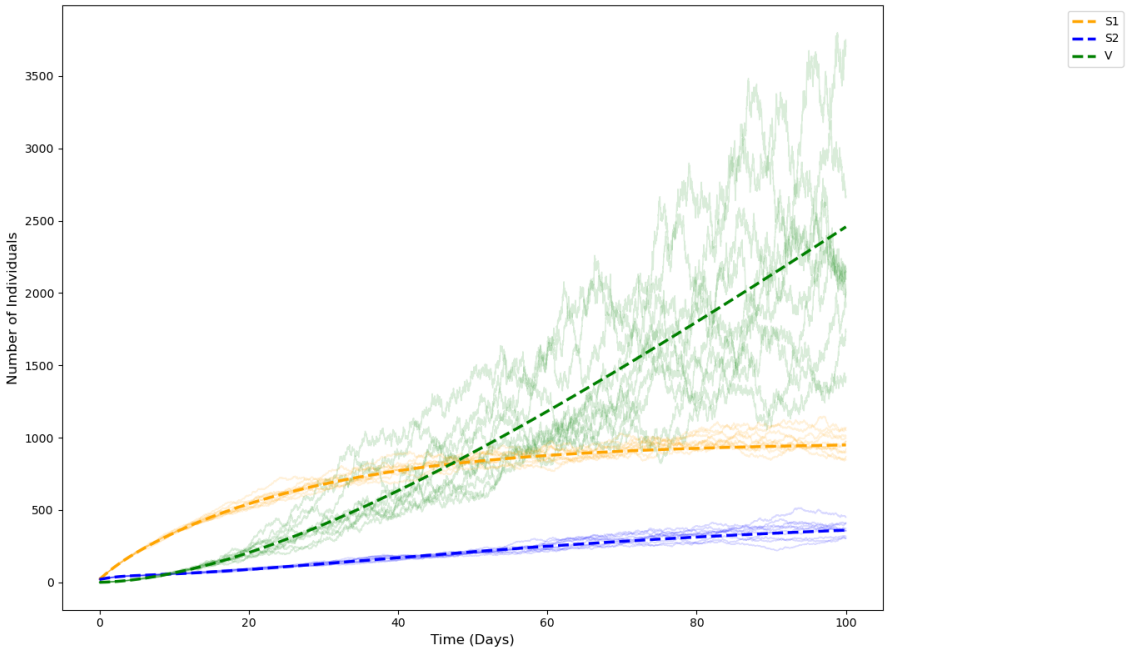


Figure 1: Stochastic plot for susceptible infant, susceptible school children and vaccinated population

Each plot features multiple jagged lines (individual stochastic realizations) alongside a smooth trend line (the deterministic baseline). From figure 1, the cluster of green curves (vaccinated population) drift farther away from the deterministic baseline compared to the cluster of orange curves (susceptible infant) and the blue curves (susceptible school children population). The high variance indicates the effect of randomness on the vaccination program. At day zero the variances between each cluster of curves reduces, and becomes very close to the deterministic baseline. After 5days each individual trajectory for the number of susceptible school children reduces and drops lower than the vaccinated population and susceptible infant population. After 40days the vaccinated population curves intercept with that of the susceptible infants. As the vaccination curves moves upward the susceptible infant curves moves downward, indicating a reduction in the number of susceptible infants.

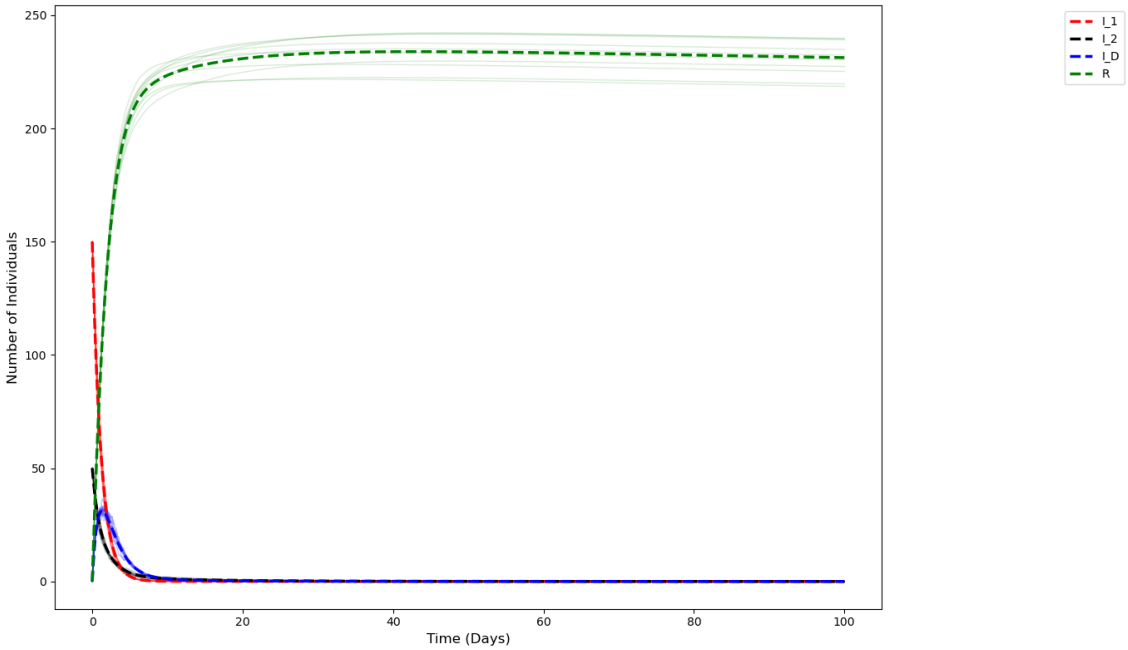


Figure 2: Stochastic plot of recovered populations (I_1, I_2,I_D) and infected populations

From Figure 2, individual trajectories for the infected classes (I_1, I_2, I_D) cluster around their deterministic baseline (the dash lines), but the recovered class have smooth curves that drifts away from its deterministic baseline. The cluster of curves for the infected classes drops to zero early, on day 10. The number of detected cases peaks at day 5, while the R curves rises quickly, maintaining a smooth trajectories all through. The steep rise in R and sharp drops in I_1, I_2, and I_D signals the end of an epidemic wave of the disease. Hence, demonstrating how randomness can cause the outbreak of Diphtheria die out.

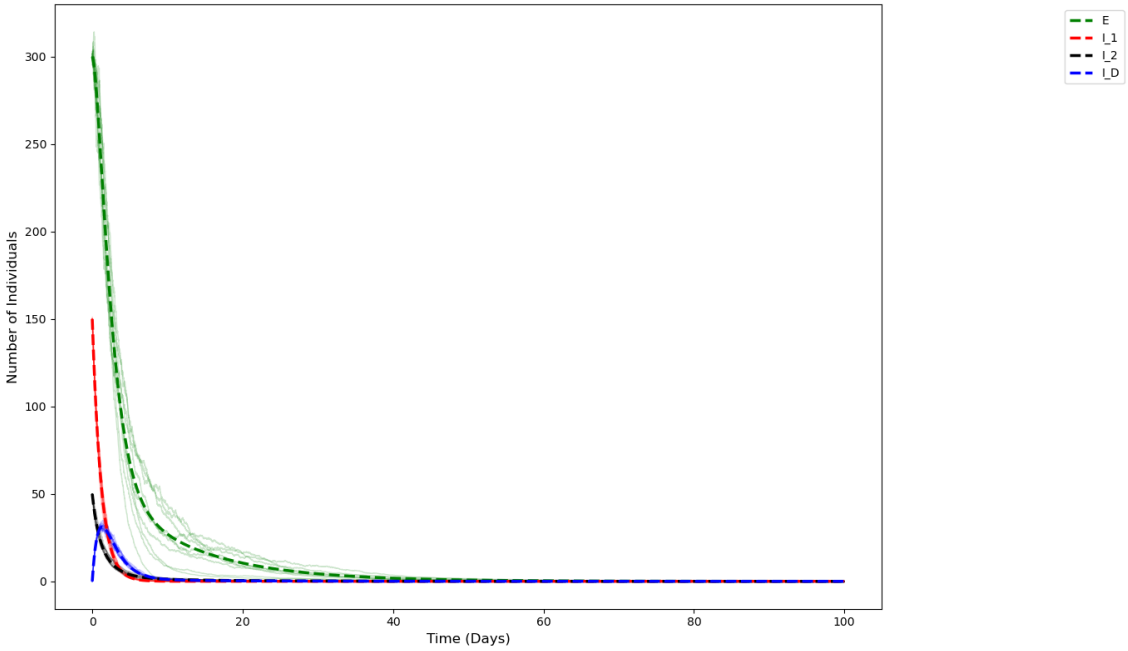


Figure 3: Stochastic plot of Exposed populations (I_1, I_2, I_D) and infected populations

The cluster of green curves (Exposed population) drops sharply from its initial value to 40 individuals by day 10 with some curves dropping to zero on day 10, while its deterministic baseline drops gradually to zero on day 40. The rate of the drop corresponds to how quickly the diphtheria epidemic spreads, combined with how fast individuals move to the infected classes. The rapid drop in the asymptomatic and symptomatic classes to zero by day 5 also indicates how fast individuals move to the detected infected class. The detected infected population (I_D) curves plateaus earlier; this indicates the

effect of random noise in detecting the disease hence suppressing the initial infection wave.

CONCLUSION

In this work, both the age-structured deterministic and stochastic model of the transmission dynamics of diphtheria infection has been developed. The model includes specific epidemiological classes such as S_1 Susceptible infant at time t (0-1years), S_2 , Susceptible school children population at time t, V, Vaccinated population at time t, E, Exposed population at time t , I_1, Asymptomatic infected population at time t, I_2., symptomatic infected population at time t, I_D., Detected infectious humans at time t(Asymptomatic and symptomatic) population through testing, R., Recovered population at time t . The research focuses on the transformation of the deterministic model into the stochastic model using the Langevin approach. The transition probabilities that capture the likelihood of events inherent in the transmission process were constructed, resulting in the transition probability table. Using the modeling procedure developed by[7], the stochastic differential equation (SDE) for the transmission dynamics of the age-structured diphtheria infection was developed. The numerical simulation of the model shows the following: The high variance indicates the effect of randomness on the vaccination program. At day zero the variances between each cluster of curves reduces, and becomes very close to the deterministic baseline. After 5days each individual trajectory for the number of susceptible school children reduces and drops lower than the vaccinated population and susceptible infant population. After 40days the vaccinated population curves intercept with that of the susceptible infants. As the vaccination curves moves upward the susceptible infant curves moves downward, indicating a reduction in the number of susceptible infants. The result shows that increased vaccination rate will lead to Diphtheria infection extinction while low vaccination rate will produce an epidemic.

From figure 2, the number of detected cases peaks at day 5, while the R curves rises quickly, maintaining a smooth trajectories all through. The steep rise in R and sharp drops in I_1, I_2, and I_D signals the end of an epidemic wave of the disease. Hence, demonstrating how randomness can cause the outbreak of Diphtheria die out.

From figure 3, The rapid drop in the asymptomatic and symptomatic classes to zero by day 5 also indicates how fast individuals move to the detected infected class. The detected infected population (I_D) curves plateaus earlier; this indicates the effect of random noise in detecting the disease hence suppressing the initial infection wave.

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