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Perturbation-stable failure and weighted dual repair for harmonic-mean transportation allocations

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Abstract

Harmonic-mean transportation rules can generate feasible schedules without providing a cost guarantee. This paper studies the rule that selects the active row or column of largest harmonic mean and allocates to its cheapest cell. An explicit positive-cost family has unbounded approximation ratio, and a perturbed family shows that the failure persists throughout a full-dimensional neighbourhood with strict deciding choices. For every route-cost perturbation of magnitude at most 0.01, the ratio is bounded below by $(M + 11.97)/13.03$ for $M \geq 32$. A weighted potential-correction problem then computes the strongest transportation dual bound obtainable by downward movement of supplied candidate potentials. Its bipartite packing dual supplies an independent repair certificate, and the resulting bound dominates a scalar correction. Computation includes 300 assignment instances, 600 perturbation tests and 120 rectangular transportation instances. The weighted correction improves the scalar bound on all rectangular cases, with a mean correction-loss ratio of approximately 0.372. The worst-case theorem concerns the specified initial rule; it does not apply to subsequent exact improvement. The study combines a perturbation-stable negative result with an auditable certificate framework. All numerical data are synthetic, and the underlying LP duality is established.

Keywords: transportation problem; harmonic mean; approximation ratio; linear programming duality; industrial distribution; reproducible computation.

1. Introduction

The transportation problem translates the distribution of a homogeneous commodity from several sources to several destinations into a linear optimization model. Its classical formulation goes back to Hitchcock [10], and its relationship with linear programming is developed in standard texts [8, 4]. A feasible shipping schedule must respect supply and demand. An optimal schedule must additionally minimize the objective among all feasible schedules. These are separate requirements: a construction rule can meet every margin and still have a very large objective gap.

Harmonic-mean methods provide one way to prioritize rows and columns. In the harmonic-mean method of Palanivel and Suganya [18] and the transportation application of Okeke [14], the rule computes harmonic means, selects the largest, and makes a maximum allocation to a minimum-cost entry. Paint production [15] and pharmaceutical production [17] demonstrate how linear programming can be applied to resource decisions, and the sensitivity terminology in [16] motivates explicit parameter-response definitions in this industrial study. None of these application settings makes a feasible transport allocation automatically optimal. A numerical solution requires either a comparison with the optimum or a certificate that no cheaper feasible solution exists.

The present study addresses a precise question: can the harmonic-mean initial-allocation rule have an objective value arbitrarily larger than the optimum? The answer is affirmative for the deterministic rule specified below. The constructed family is small enough to inspect by hand, contains no zero costs, and has balanced unit supply and demand. The proof therefore does not depend on large industrial instances, unusual unbalanced margins, or numerical rounding. The second contribution is a practical certification framework based on the transportation dual. Its underlying duality is classical; the contribution here is its explicit use to audit harmonic-mean allocations and its application to the constructed family.

The study does not claim that every harmonic-mean variant has the same behaviour. Tie-breaking, recomputation and zero-cost conventions can define different algorithms. It also does not claim that MODI or stepping-stone improvement fails. Such improvement changes the initial schedule and is a distinct optimization stage [7]. The result concerns the initial heuristic alone. The benchmark is exploratory and synthetic; the proof, rather than average numerical performance, establishes the worst-case conclusion.

The strengthened analysis adds two linked results. First, a full-dimensional perturbed family retains the bad shipment with strict deciding choices. Second, a weighted covering problem repairs candidate dual potentials with the smallest objective loss within a specified downward-correction class. A larger rectangular study checks the repair against independent primal and packing-dual optimizers.

2. Model and deterministic allocation rule

Let $m, n \geq 1$ denote the number of sources and destinations. Let $a_i > 0$ be available supply at source i , $b_j > 0$ the required quantity at destination j , and $c_{ij} > 0$ the unit shipping cost. The

decision variable $x_{ij} \geq 0$ is the quantity shipped on route (i, j) . The balanced problem is

$$Z^* = \min_X \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}, \quad (1)$$

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, \dots, m, \quad (2)$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, \dots, n. \quad (3)$$

Balance means $S = \sum_i a_i = \sum_j b_j$. All costs are finite and strictly positive in the analysis. A dummy zero-cost destination is therefore outside the stated algorithm unless a separate zero-cost convention is supplied. Introducing artificial positive dummy costs changes the objective and must not be hidden.

At an iteration, R and J denote active sources and destinations with positive remaining margins. The active harmonic means are

$$H_i = \frac{|J|}{\sum_{j \in J} 1/c_{ij}}, \quad G_j = \frac{|R|}{\sum_{i \in R} 1/c_{ij}}. \quad (4)$$

The rule selects the largest of all H_i and G_j . Ties between rows and columns favour rows; ties within an axis favour the smallest original index. Within the selected line, choose the minimum-cost active cell, again breaking ties by the smallest index on the other axis. Allocate the smaller remaining margin to that cell. Subtract the allocation from both margins, remove exhausted rows and columns, recompute all means, and continue. Removing both exhausted margins is legitimate for feasibility; degeneracy matters only if a later simplex implementation requires an explicit basis.

Proposition 2.1 (Feasibility and termination). *For positive balanced margins and a complete finite cost matrix, the stated rule terminates with a feasible allocation after at most $m + n - 1$ positive allocations.*

Proof. Each allocation is nonnegative, does not exceed either current margin, and reduces source and destination totals by the same amount. Therefore balance and nonnegative remaining margins are invariant. At least one active line is exhausted at each allocation. The last allocation exhausts both the final source and final destination. Starting with $m + n$ active lines gives the claimed bound. A positive balanced residual cannot have an empty set of sources or destinations, so the procedure cannot stop before every margin is satisfied. $\square$

The proposition gives no objective guarantee. The harmonic mean measures the magnitude of costs in a line; it does not measure the opportunity cost of consuming a source or destination. That missing information is what the counterexample exploits.

3. An explicit unbounded-gap family

Consider unit supply and unit demand with

$$C(M) = \begin{pmatrix} 3 & M & M \\ 7 & M & 8 \\ 1 & 2 & 5 \end{pmatrix}, \quad M \geq 32. \quad (5)$$

Unit margins make this a transportation problem whose integral extreme points are permutation matrices. The available low-cost assignment is $(1, 1), (2, 3), (3, 2)$, but the heuristic initially consumes destination 1 using source 2.

Theorem 3.1 (Exact behaviour and unbounded ratio). *For every $M \geq 32$, the deterministic harmonic-mean rule applied to (5) returns $(2, 1), (1, 2), (3, 3)$ with cost $Z_H = M + 12$. The exact optimum is $Z^* = 13$. Hence*

$$\frac{Z_H}{Z^*} = \frac{M+12}{13}, \quad Z_H - Z^* = M - 1, \quad (6)$$

and no finite constant bounds Z_H/Z^* over all positive balanced transportation instances.

Proof. The initial row means are

$$H_1 = \frac{9M}{M+6} < 9, \quad H_2 = \frac{168M}{15M+56} > 10, \quad H_3 = \frac{30}{17} < 2.$$

The inequality $H_2 > 10$ follows from $18M > 560$, valid when $M \geq 32$. The column means satisfy

$$G_1 = \frac{63}{31} < 3, \quad G_2 = \frac{6M}{M+4} < 6, \quad G_3 = \frac{120M}{13M+40} < \frac{120}{13} < 10.$$

Thus row 2 is uniquely selected. Its cheapest entry is 7 in column 1, since $7 < 8 < M$, so one unit is allocated to $(2, 1)$. The remaining matrix, indexed by rows 1 and 3 and columns 2 and 3, is

$$\begin{pmatrix} M & M \\ 2 & 5 \end{pmatrix}.$$

Row 1 has harmonic mean M , while the other row has mean $20/7$ and each column has mean below 10. Row 1 is uniquely selected. Its equal entries are resolved by the specified index rule, selecting column 2. The last allocation is $(3, 3)$. The total is $7 + M + 5 = M + 12$.

The six permutation costs are $M + 8$, 13, $M + 12$, $M + 9$, $M + 9$ and $2M + 1$. Their minimum is 13 for $M \geq 32$. Alternatively, the dual certificate below proves the optimum without relying on permutation integrality. Finally, the ratio tends to infinity as $M \rightarrow \infty$, establishing the absence of a uniform multiplicative guarantee. $\square$

Remark 3.2 (Dependence on the final cost tie). If the second-stage equal-cost tie is resolved in favour of column 3 instead, the allocation costs $M + 9$, which also gives an unbounded ratio. Thus the unbounded conclusion survives either admissible choice in that tie, although the exact formula $M + 12$ uses the specified convention. The first selection has no tie.

3.1. Replication and cost-scale implications

Replicate the unit margins by a common quantity $q > 0$: set every supply and demand equal to q . The same line sequence is obtained because harmonic means depend on the cost matrix rather than on q . Both objective values are multiplied by q , and the ratio is unchanged. The family therefore represents arbitrary shipment quantities, not merely one physical unit.

For a fixed instance with costs between $c_{\min} > 0$ and $c_{\max}$, every feasible schedule satisfies $Sc_{\min} \leq Z \leq Sc_{\max}$. Consequently $Z_H/Z^* \leq c_{\max}/c_{\min}$. This instance-dependent bound is compatible with the theorem: in the family, $c_{\max} = M$ grows while $c_{\min} = 1$. Multiplying all costs by a positive scalar preserves the algorithm and the ratio, but adding a common constant to all costs can alter harmonic-mean priorities. Although the optimal allocation of a balanced fixed-mass problem is unaffected by such an additive shift, a heuristic need not share that invariance.

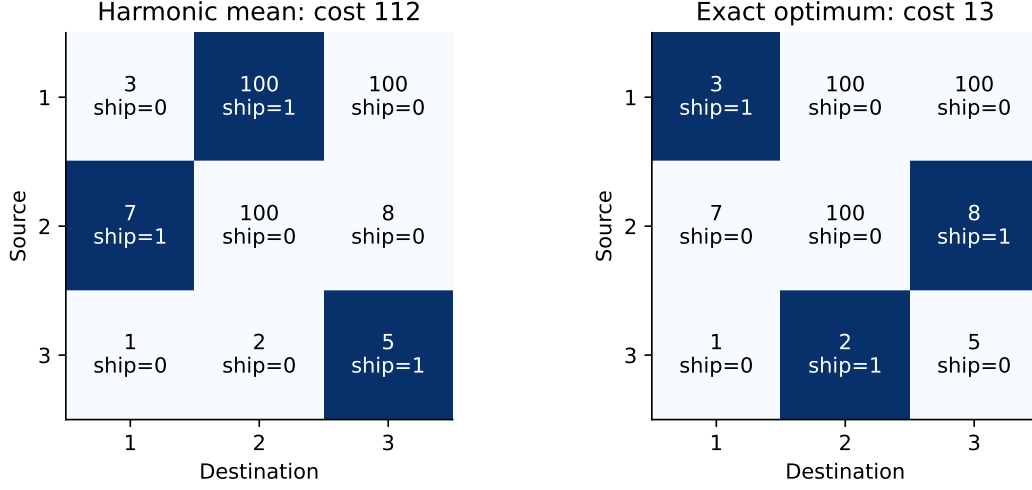


Figure 1: Route costs and shipping allocations for $M = 100$. The harmonic-mean initial cost is 112, while the certified optimum is 13. The instance is synthetic.

4. Dual certificates and a computable gap audit

The transportation dual introduces unrestricted source potentials u_i and destination potentials v_j :

$$\max_{u,v} D(u,v) = \sum_i a_i u_i + \sum_j b_j v_j, \quad u_i + v_j \leq c_{ij}. \quad (7)$$

The potentials are accounting values for the supply and demand constraints, not route shipments. Standard linear programming theory supplies weak and strong duality [8, 4]. The next result is included with proof because it provides a direct audit of a heuristic output.

Theorem 4.1 (Optimality certificate). *For a feasible allocation X and dual-feasible (u, v) ,*

$$Z(X) - D(u, v) = \sum_{i,j} x_{ij} (c_{ij} - u_i - v_j) \geq 0. \quad (8)$$

If every positive allocation has zero reduced cost $c_{ij} - u_i - v_j$, then X is optimal and $Z(X) = D(u, v) = Z^$.*

Proof. Expand the right-hand side and use the row and column margin equations to replace $\sum_j x_{ij}$ by a_i and $\sum_i x_{ij}$ by b_j . Every term is nonnegative by feasibility of the primal and dual. Zero reduced cost on all positive cells gives equality, so weak duality prevents any cheaper feasible allocation. $\square$

For the family, set $u = (3, 7, 0)$ and $v = (0, 2, 1)$. Then

$$C(M) - (u_i + v_j)_{ij} = \begin{pmatrix} 0 & M-5 & M-4 \\ 0 & M-9 & 0 \\ 1 & 0 & 4 \end{pmatrix} \geq 0.$$

The dual objective is $3 + 7 + 0 + 0 + 2 + 1 = 13$. The allocation $(1, 1), (2, 3), (3, 2)$ uses only zero reduced-cost entries and also costs 13. This certifies the optimum for all $M \geq 32$. The heuristic gap is exposed as the sum of reduced costs on its selected cells: $0 + (M - 5) + 4 = M - 1$.

Proposition 4.2 (Repair of arbitrary potentials). *Given any finite (u, v) , let*

$$\delta = \max\{0, \max_{i,j}(u_i + v_j - c_{ij})\}, \quad \hat{v}_j = v_j - \delta.$$

Then $(u, \hat{v})$ is dual feasible. Set $L = \max\{Sc_{\min}, D(u, v) - S\delta\}$. Every feasible X satisfies

$$0 \leq Z(X) - Z^* \leq Z(X) - L, \quad 1 \leq Z(X)/Z^* \leq Z(X)/L. \quad (9)$$

Proof. The definition of δ implies $u_i + v_j - \delta \leq c_{ij}$. Lowering every destination potential reduces the dual objective by $\delta \sum_j b_j = S\delta$. Both this repaired objective and $Sc_{\min}$ are lower bounds on Z^* . Their maximum is positive, so subtraction and division yield the stated inequalities. $\square$

The bound is useful even if the potentials are not optimal. A small audited gap certifies a good feasible schedule; a large bound is inconclusive and should trigger improvement or exact optimization. Potentials fitted to occupied cells can fail on unoccupied routes. For that reason, checking all dual inequalities is essential. An acyclic basis permits potentials to be propagated along occupied edges, but degeneracy or disconnected support must be handled explicitly by any basis-based implementation.

5. Computational protocol and results

All experiments use the included Python source. The adversarial family is evaluated at $M \in \{32, 50, 100, 500, 1000, 10000\}$. The heuristic allocation is checked against the exact assignment solver and the analytic formula. The general benchmark draws 100 independent matrices for each size $n \in \{3, 5, 8\}$, with integer entries sampled uniformly from 1 to 100 using NumPy generator seed 20261008. Every margin is one. Each matrix is evaluated by four initial rules: harmonic mean (HM), Vogel approximation (VAM), least cost (LC) and north-west (NW). VAM uses the difference of the two smallest active costs; a single remaining entry uses its own cost as the score. All methods have deterministic index ties.

The exact assignment optimum is computed by SciPy's linear-sum-assignment routine. This benchmark is a special case of transportation, so it must not be described as a general capacitated-distribution field study. Reported quantities are objective ratios and counts of optimal initial allocations. Runtime comparisons are omitted because machine-dependent timing was not designed as a controlled experiment.

Table 1: Exact adversarial-family results under the stated tie rule.

M	Heuristic	Optimum	Ratio	Gap
32	44	13	3.3846	31
50	62	13	4.7692	49
100	112	13	8.6154	99
500	512	13	39.3846	499
1000	1012	13	77.8462	999
10000	10012	13	770.1538	9999

HM achieves the optimum on 61, 50 and 24 of the 100 matrices at sizes 3, 5 and 8, respectively. Its mean ratios are approximately 1.0890, 1.1333 and 1.2117. VAM has lower mean

Table 2: Seeded synthetic benchmark: 100 instances at each size.

Size	Method	Mean ratio	Maximum ratio	Optimal count
3	HM	1.0890	1.7556	61
3	VAM	1.0119	1.3947	90
3	LC	1.1577	4.4783	63
3	NW	1.7724	4.6512	22
5	HM	1.1333	2.3404	50
5	VAM	1.0416	1.5488	68
5	LC	1.2402	2.9302	31
5	NW	2.4844	11.7857	4
8	HM	1.2117	1.9493	24
8	VAM	1.0456	1.4297	56
8	LC	1.3144	3.0882	15
8	NW	3.1783	6.4684	0

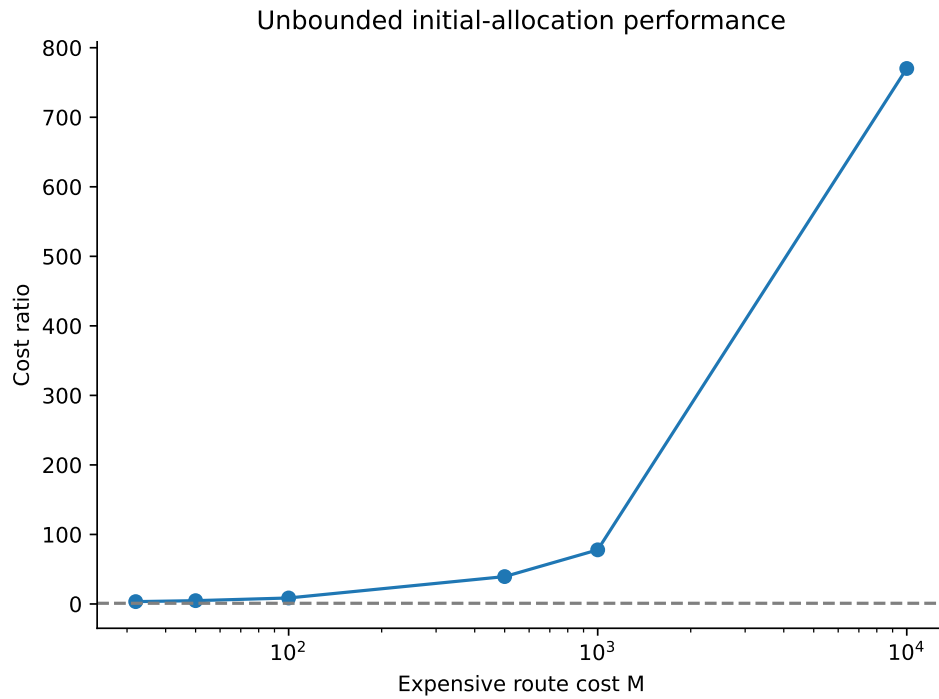


Figure 2: The exact ratio $(M + 12)/13$ grows without bound. A bounded random benchmark cannot establish a worst-case guarantee.

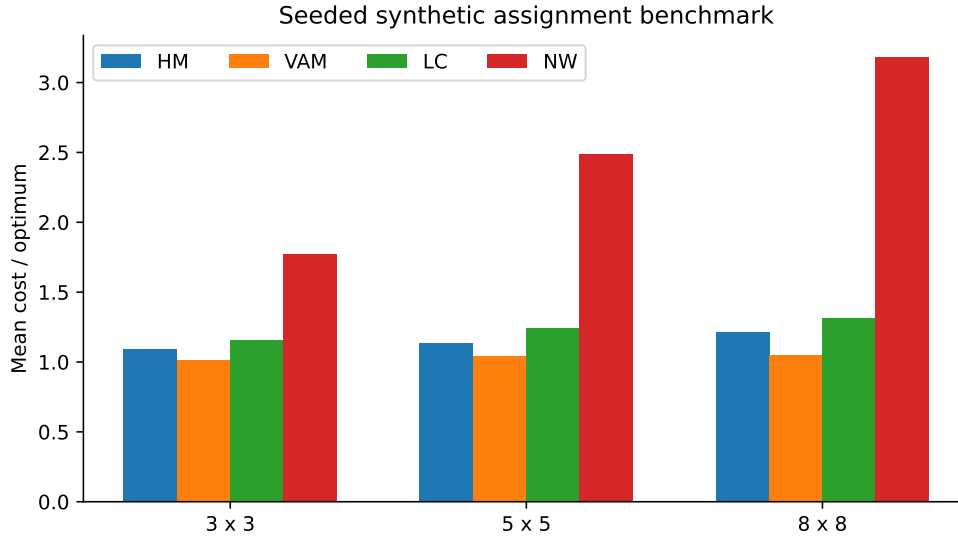


Figure 3: Mean initial cost ratios for the 300 seeded synthetic matrices. No MODI, stepping-stone or simplex improvement is included in these ratios.

ratios in this particular sample. This observation is not a stochastic dominance result and does not show that VAM is universally superior. LC and NW provide transparent baselines. The adverse family is deliberately selected to demonstrate a theorem; it is not part of the random sample and must not be blended into an average without disclosure.

6. Industrial dispatch interpretation and implementation

Interpret the three sources as manufacturing plants and the three destinations as customer warehouses. Route costs are normalized charges, and M represents an expensive eligible route. The instance identifies no real plants or measured costs. It illustrates why a local dispatch priority can destroy a cheaper global allocation.

Production and distribution are linked but distinct. The paint-production and medication-manufacturing applications [15, 17] motivate an upstream resource-allocation layer. Here, every shipment is a generic commodity quantity; no biological variables or patient outcomes are modelled. If production quantities are decision variables rather than fixed supplies, the complete production–distribution model must include both sets of constraints. Solving the two stages independently can miss a better coupled plan.

An implementation should validate balance and positive costs, construct the initial allocation, verify every margin and its objective, and obtain a dual lower bound or solve the exact model. When improvement is used, both initial and final costs should be reported. Numerical tolerances must reflect cost and quantity scales. Zero-cost routes need an explicit harmonic-mean convention.

7. Related methods and the four-paper foundation

The four selected works define a coherent production–distribution–demand chain. Harmonic-mean transportation [14] supplies the initial dispatch rule. Paint production [15] and medication production [17] supply the resource-constrained upstream allocation setting. Coronavirus sensitivity [16] supplies a motivating demand context, rather than evidence that any particular

hospital demand is known. This paper contributes a certificate audit of the distribution stage; it does not reproduce the numerical datasets of those four works.

Assignment algorithms have a substantial exact-method literature, including Kuhn's Hungarian approach [11], Munkres' assignment and transportation algorithms [12], and the systematic assignment treatment of Burkard et al. [6]. Their relevance is direct: unit-margin experiments should be checked against an exact assignment method. The benchmark's SciPy routine is an implementation of an exact assignment solver; it is not claimed to be a literal implementation of the original historical algorithm.

Ford and Fulkerson [9] and Ahuja et al. [1] place transportation in the larger network-flow framework. Integrality and polyhedral reasoning are treated by Schrijver [20]. They explain why a continuous balanced unit-margin LP has an integral optimum. Standard LP texts [8, 4, 22], convex analysis [5], and numerical optimization [13] support a distinction between mathematical dual feasibility and a floating-point tolerance check. The certificate must verify every reduced cost; merely meeting a solver stopping criterion is insufficient for a hand-verifiable exact statement.

Stepping-stone optimization [7] improves a starting schedule and therefore changes the question addressed by the counterexample. The classical transportation formulation [10] does not prescribe the harmonic-mean initialization. The present no-ratio result concerns that initialization before any exact improvement.

7.1. When uncertain costs enter the audit

Suppose each route cost is independently uncertain in $[\underline{c}_{ij}, \bar{c}_{ij}]$. For a fixed nonnegative shipping plan, the maximum objective is $\sum \bar{c}_{ij}x_{ij}$, so static box-robust transportation reduces to a transportation problem with upper-endpoint costs. This follows directly by maximizing each nonnegative term. Set-inclusive uncertainty [21], general robust optimization [2], and budgeted uncertainty [3] provide established alternatives when simultaneous worst-case endpoints are too conservative. The upper-cost reduction does not imply that the nominal harmonic-mean plan is robustly optimal. It can be audited with the same dual machinery applied to $\bar{C}$.

If costs are measured rather than exact, the certificate can instead use known lower endpoints to bound the true optimum. A nominal dual inequality alone cannot guarantee validity for every admissible cost. This extension connects the proof to uncertain industrial logistics while preserving the distinction between an uncertainty set and empirical evidence about that set.

7.2. Positioning against the closest literature

The comparison with published harmonic-mean methods is concrete. Palanivel and Suganya [18] use the same largest-line-mean and cheapest-cell construction and infer global optimality from their examples. The family above shows that such a universal inference is false for that initial construction, and Theorem 8.2 removes dependence on deciding ties. Okeke's application [14] motivates the audited rule. Pokalas et al. [19] develop a different initial-allocation heuristic and test a small set of examples; that paper does not establish the minimum-loss correction of specified candidate potentials.

The proposed contribution is therefore the strict-choice perturbation-stable failure result, paired with the restricted weighted correction and its empirical audit. The covering and packing duality is standard. An unsuccessful search for an identical construction cannot establish worldwide priority. Neither faster solving than exact transportation optimization nor a new general duality principle is claimed.

8. Perturbation-stable failure without allocation ties

The preceding construction specifies a tie rule. A stronger objection is obtained by removing ties at every decision that determines the bad allocation. Let

$$\widehat{C}(M) = \begin{pmatrix} 3 & M & M+1 \\ 7 & M & 8 \\ 1 & 2 & 5 \end{pmatrix}, \quad M \geq 32. \quad (10)$$

Each supply and demand is one. The relevant harmonic-mean line rule remains exactly the rule defined in Section 2; the result is not asserted for differently defined harmonic-mean heuristics.

Lemma 8.1 (Lipschitz bound for positive harmonic means). *For $h(z) = k/(\sum_{j=1}^k 1/z_j)$ on the positive orthant,*

$$|h(z) - h(w)| \leq k\|z - w\|_\infty.$$

Proof. Write $S = \sum_j 1/z_j$. Then $\partial_j h = k/(S^2 z_j^2)$ and $\sum_j \partial_j h = k \sum_j (1/z_j)^2 / S^2 \leq k$. Integrating the gradient along the positive line segment between z and w gives the bound. $\square$

Theorem 8.2 (An open set of arbitrarily poor allocations). *For every $M \geq 32$ and every matrix E satisfying $\max_{ij} |E_{ij}| \leq \varepsilon = 0.01$, the harmonic-mean rule on $\widehat{C}(M) + E$ selects $(2, 1)$, then $(1, 2)$, then $(3, 3)$. The first two deciding line maxima and their cell minima are strict; the final shipment is forced. Its approximation ratio satisfies*

$$\frac{Z_H}{Z^*} \geq \frac{M + 12 - 3\varepsilon}{13 + 3\varepsilon}. \quad (11)$$

Consequently, bounded additive perturbations and changes in tie-breaking do not restore a finite worst-case approximation factor.

Proof. Initially row 2 has harmonic mean $168M/(15M + 56) > 10$. Row 1 has mean less than 9, column 1 has mean $63/31$, column 2 has mean less than 6, column 3 has mean $120(M + 1)/(13M + 53) < 120/13$, and row 3 has mean $30/17$. Thus the winning score exceeds the runner-up by more than $10 - 120/13 = 10/13$. Each score changes by at most 3ε , so this strict winner survives. Its cheapest cell is uniquely 7, separated from 8 by one, and it too survives the perturbation.

After row 2 and column 1 are deleted, row 1 has score $2M(M + 1)/(2M + 1) > M$. The other three line scores are less than 3, 4 and 10, respectively. A change of at most 2ε per score cannot close this gap. Its minimum is M rather than $M + 1$, again with a strict gap of one. The final allocation is forced. Its cost is at least $M + 12 - 3\varepsilon$. The feasible permutation $(1, 1), (2, 3), (3, 2)$ costs at most $13 + 3\varepsilon$, so the optimum is no larger. All entries remain positive, establishing the ratio bound. $\square$

The theorem strengthens the original single-family calculation by identifying a full-dimensional neighbourhood of failures. It does not infer frequency under any empirical distribution of industrial costs. A method can work well on a selected benchmark and still have an unbounded worst-case ratio. Conversely, a worst-case failure does not measure its usual operating cost.

9. Minimum-loss downward repair of transportation potentials

The scalar repair in Section 4 pays for the largest dual violation at every unit shipped. Uneven violations can be repaired more selectively. Let a, b be nonnegative balanced supply and demand vectors, with total mass $Q > 0$, and let u, v be arbitrary candidate potentials. Define $g_{ij} = \max(0, u_i + v_j - c_{ij})$ and solve

$$W = \min_{p, q \geq 0} \{a^T p + b^T q : p_i + q_j \geq g_{ij} \forall i, j\}. \quad (12)$$

This is a weighted covering LP on the bipartite violation graph. Only positive-violation edges impose nontrivial constraints. It is an auxiliary certificate problem; it is not an improvement to the shipment itself.

Theorem 9.1 (Best downward repair and a certified stopping rule). *Let $D_0 = a^T u + b^T v$. Then $L = D_0 - W$ is the largest dual lower bound attainable by componentwise downward corrections of the specified u, v . For every feasible allocation X ,*

$$0 \leq C : X - Z^* \leq C : X - L. \quad (13)$$

If $\delta = \max_{ij} g_{ij}$, then $W \leq Q\delta$. In particular, the weighted bound is no weaker than the scalar-repair bound.

Proof. The corrected potentials $u - p, v - q$ are feasible exactly when $p_i + q_j \geq u_i + v_j - c_{ij}$, which is equivalent to (12) for nonnegative corrections. Their objective is $D_0 - a^T p - b^T q$, establishing the optimality within this restricted repair class. Weak duality yields (13). The feasible choice $p_i = \delta, q_j = 0$ costs $Q\delta$, so W cannot be larger. $\square$

The repair class matters: an unrestricted reoptimization of all dual potentials can supply a stronger bound. Here the candidate values are preserved wherever their downward movement is unnecessary. Gauge changes $u \mapsto u + \gamma \mathbf{1}$, $v \mapsto v - \gamma \mathbf{1}$ leave both g and D_0 unchanged, because the model is balanced. Therefore the repair is invariant to the arbitrary potential origin.

The dual of (12) is

$$W = \max_{F \geq 0} \left\{ \sum_{ij} g_{ij} F_{ij} : \sum_j F_{ij} \leq a_i, \sum_i F_{ij} \leq b_j \right\}. \quad (14)$$

A feasible F with an equal objective independently certifies the repair optimum. This is a capacitated bipartite packing interpretation of standard LP duality [1, 20, 5]. It is distinct from a dual-named allocation heuristic such as [19], which proposes an initial transportation solution rather than the minimum-loss potential correction above.

For an inexpensive preliminary audit, set $p_i = \max_j g_{ij}, q = 0$, or $p = 0, q_j = \max_i g_{ij}$. The cheaper of those bounds is also dominated by W . A practical workflow tests these cheap bounds first, solves the weighted repair if needed, and invokes an exact shipment optimizer if the certified gap exceeds the accepted accounting tolerance. No speed advantage over an exact optimizer is assumed.

9.1. Expanded experiments and interpretation

The additional script `advanced_experiments.py` tests 600 bounded perturbations of (10), checks that every allocation follows the theorem, and compares scalar, row-only, column-only

and weighted corrections on 120 rectangular transportation instances. Sizes include 4×6 , 8×12 and 12×18 . Supplies and demands are positive, fractional and independently balanced; costs are drawn from bounded integer and lognormal families. A transportation LP supplies the exact comparison optimum. Candidate row potentials are the largest cost on each row's positive-shipment support and column potentials are zero. This definition is reproducible even when support is disconnected.

Every reported repair bound is checked against the exact optimum, its corrected inequalities, and the independent packing dual. Results therefore assess certificate tightness, not a claimed reduction in shipping cost. The archive includes all instance-level records and a figure comparing certificate gaps. Runtime measurements are reported as local observations only. The small assignment benchmark and the larger rectangular study answer different questions and are retained separately.

Table 3: Expanded synthetic study: measured numerical summaries.

Metric	Result
Perturbation tests	600
Rectangular instances	120
Strictly tighter weighted certificates	120
Mean weighted/scalar correction loss	0.371731
Maximum repair primal–dual gap	1.36424e-12

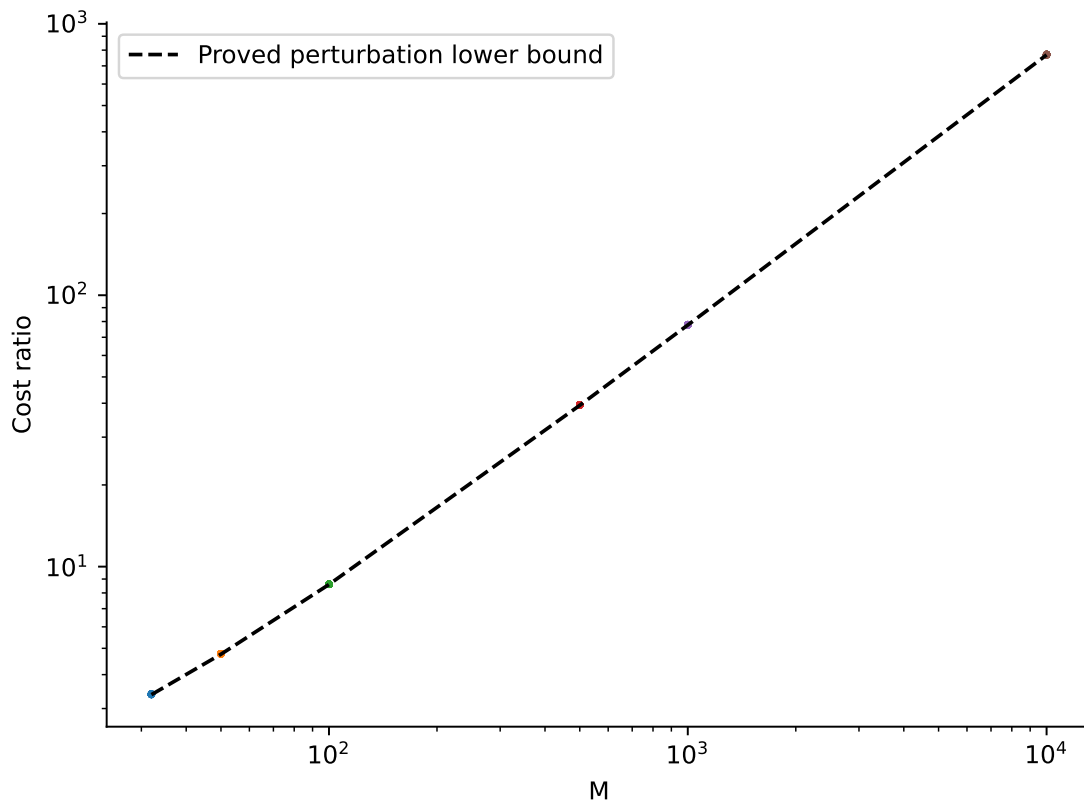


Figure 4: The proved failure survives 600 bounded cost perturbations. Points are synthetic samples; the dashed curve is the analytic lower bound.

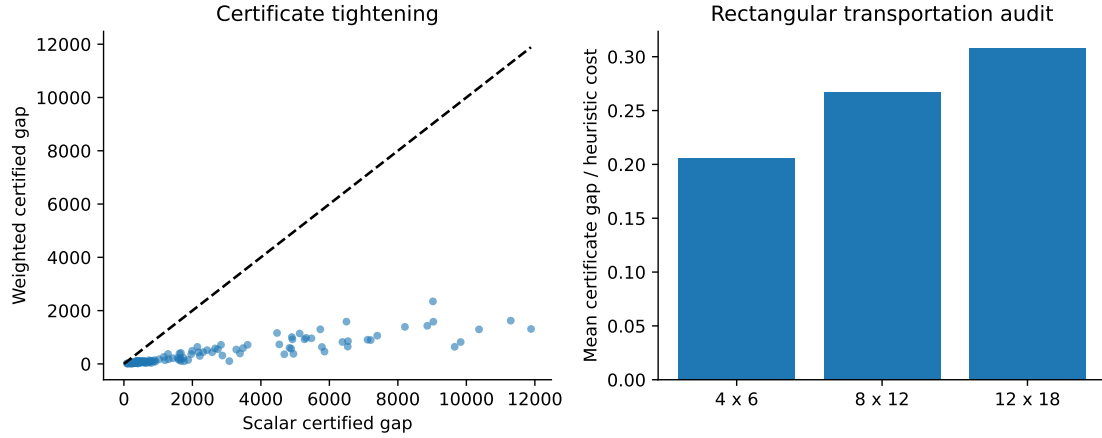


Figure 5: Weighted and scalar cost-gap certificates for rectangular transportation. Both curves audit the original schedule without changing it.

10. Limitations and conclusion

The largest-harmonic-mean initial rule has an unbounded cost ratio, and the failure persists under uniformly bounded entry perturbations with strict deciding choices. A weighted downward potential repair gives a transparent gap certificate and dominates a uniform correction. Its independent packing dual separates correctness of the certificate from quality of the initial shipment. The synthetic experiments confirm these mathematical checks without establishing field performance. The appropriate industrial use is an initial schedule accompanied by an audited gap or subsequent optimization.

The theorem addresses the explicitly defined initial rule and positive balanced costs. It does not cover all harmonic-mean variants, zero-cost conventions, fixed charges or integer-batch shipping. Floating-point checks are numerical verification rather than exact interval-arithmetic proof. No real distribution dataset is used.

A. Reproducibility and verification checklist

The archive contains the main LaTeX file and its included text files. It also contains vector and raster graphics, the experiment script, CSV outputs and machine-readable results. Execute `bash reproduce.sh` to run both experiment scripts and compile the manuscript twice. Python requires NumPy, SciPy and Matplotlib. The script checks the family objectives directly. A manual verification can enumerate all six permutations and check the nine dual inequalities. These small checks are independent of the heuristic code and protect the central claim against implementation mistakes.

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