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Volatility Sensitivity of Deep In-The-Money European Call Options: A Black-Scholes and Non-Central F Analysis

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ABSTRACT

This study investigates the sensitivity of European call option prices to volatility under the Black-Scholes framework for deep in-the-money contracts. Using the stock quantity parameter values to compute, the results show a monotonic and convex increase in call value with volatility, with total increases of \$2.33 and \$1.84 for $S_0 = 60$ and $S_0 = 70$, respectively. A Fisher non-central F analysis confirms that volatility explains a statistically significant proportion of price variation, with a strong effect size after controlling for the level of the initial stock price. The findings underscore the critical role of vega for deep in-the-money options and the importance of accurate volatility estimation in high-rate environments.

Keywords: European Call Option; Deep-in-the-money; Volatility Sensitivity; Vega; Non-Central F Distribution; ANOVA; Option Pricing; Interest Rate Risk; Statistical Significance.

1. Introduction

In financial markets, investors and financial analysts are generally interested in how to maximize profit over particular trading days, that is, the changes in the price of goods and services. Therefore, modelling the behaviour of a stock exchange market can be made through its relative change of unstable market variables in time so as to predict stock price fluctuation, advise investors, and support corporate owners who are working out convenient ways to do business by issuing stocks in their corporations. Option pricing under uncertainty remains central to financial risk management. The Black-Scholes model provides a closed-form solution where volatility is the only unobservable input and enters through the vega term [5, 4, 16]. While vega is well understood for at-the-money options, its behaviour for deep in-the-money options under high interest rates is less documented empirically. In emerging markets such as Nigeria, with interest rate of 20%, the time value component is compressed, making the role of volatility more pronounced.

Several scholars have written extensively on Black-Scholes option pricing and related stochastic models. Amadi and Umoh [1] modified the Black-Scholes model to assume a probability which measures the risk-free interest rate of the underlying asset for call and put options. Mandah, Charles and Amadi [2] considered closed-form option prices and their approximate prices by exploring covariance matrix solutions and eigenvalue properties for stock option prices. Kwok [3], Hull [15, 4], Wilmott, Howison and Dewynne [17], and Higham [16] provide the mathematical foundations of derivative valuation. Black and Scholes [5] first demonstrated the valuation relevance of options through a no-arbitrage framework that leads to the partial differential equation governing option price dynamics.

Additional work has examined implied volatility and the risk-free rate [6], Hermite polynomial representations of European option valuation [7], Laplace-transform solution techniques for the Black-Scholes terminal value problem [8], time-varying parameter solutions [9], stochastic stability of stock market prices [10, 11], and Crank-Nicolson finite difference valuation [12]. These studies support the continuing relevance of Black-Scholes modelling, but few isolate volatility sensitivity in deep in-the-money calls using both exact computation and inferential statistics.

Practitioners often assume that deep in-the-money options have negligible vega because intrinsic value dominates. However, in high-rate environments, the present value of the strike price falls, increasing the option's exposure to volatility. There is limited quantitative evidence on the magnitude and statistical significance of this volatility effect for deep in-the-money calls. Without it, risk managers may under-hedge vega. This paper quantifies and statistically validates the impact of volatility on Black-Scholes call prices for deep in-the-money options by combining exact Black-Scholes computation with graphical analysis and an ANOVA-based non-central F interpretation.

2. Preliminaries

Here we present definitions and foundational concepts for the mathematical finance model.

Definition 1. Probability space. A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω represents the sample space, $\mathcal{F}$ represents a collection of subsets of Ω, and $\mathbb{P}$ is the probability measure defined on each event $A \in \mathcal{F}$. The collection $\mathcal{F}$ is a σ-algebra such that $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under countable unions and complements [14, 16].

Definition 2. Normal distribution. A normal distribution is central in probability theory and is usually used for modelling asset returns. A standard normal cumulative distribution function is defined as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt. \tag{1}$$

A normal distribution is symmetric, and the symmetry relation is given by

$$\Phi(x) = 1 - \Phi(-x). \tag{2}$$

Definition 3. A σ-algebra is a set $\mathcal{F}$ of subsets of Ω with the following axioms:

$$\emptyset, \Omega \in \mathcal{F}, \tag{3}$$

$$\text{if } A \in \mathcal{F}, \text{ then } A^c \in \mathcal{F}, \tag{4}$$

$$\text{if } A_1, A_2, \ldots \in \mathcal{F}, \text{ then } \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}. \tag{5}$$

Clearly,

$$A^c = \Omega - A$$

is the complement of A.

Definition 4. If $\mathcal{F}$ is a σ-algebra in Ω, then Ω is called a measurable space and the members of $\mathcal{F}$ are called measurable sets in Ω.

Definition 5. Let (Ω, M) be a measurable space. A map $\mu : M \to [0, \infty]$ is called a measure provided that

$$\mu(\emptyset) = 0, \tag{6}$$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n), \tag{7}$$

for countable disjoint measurable sets A_n.

Definition 6. A Gaussian process is a stochastic process whose finite-dimensional probability distributions are all Gaussian.

Definition 7. Random walk. A random walk is a stochastic sequence $\{S_n\}$ with $S_0 = 0$, defined by

$$S_n = \sum_{k=1}^{n} X_k, \tag{8}$$

where X_k are independent and identically distributed random variables.

Definition 8. Stochastic differential equation. Let $S(t)$ be the price of a risky asset at time t, α be an expected rate of return on the stock, and dt be a relative change during trading days. The stock follows a random walk governed by the stochastic differential equation

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW_t, \tag{9}$$

where α is drift, σ is volatility, and W_t is Brownian motion on a probability space [16, 17].

Theorem 2.1 (Ito's formula). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space and let $X(t)$ be an adapted stochastic process satisfying

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t). \tag{10}$$

For $u = u(t, X(t)) \in C^{1,2}([0,1] \times \mathbb{R})$,

$$du(t, X(t)) = \left[\frac{\partial u}{\partial t} + g\frac{\partial u}{\partial x} + \frac{1}{2}f^2\frac{\partial^2 u}{\partial x^2}\right] dt + f\frac{\partial u}{\partial x}dW(t). \tag{11}$$

Using Ito's formula, the asset price solution may be written as

$$S(t) = S_0 \exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}, \quad t \in [0,1].$$

The dynamics of option pricing is given by the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{12}$$

The analytic formula for the price of a European call option is given by [5, 15, 16]

$$\begin{aligned} C &= SN(d_1) - Ke^{-rT}N(d_2), \\ d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned} \tag{13}$$

where C is the price of a call option, S is the price of the underlying asset, K is the strike price, r is the riskless rate, T is time to maturity, σ is volatility, and N is the cumulative normal distribution.

Similarly, the formula for the price of a European put option is

$$\begin{aligned} P &= Ke^{-rT}N(-d_2) - SN(-d_1), \\ d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned} \tag{14}$$

where P is the price of a put option and the other parameters retain their meanings [15, 4, 16, 17].

3. Materials and Methods

This study evaluates the sensitivity of European call option prices to volatility for two deep in-the-money stock price levels, $S_0 = 60$ and $S_0 = 70$. The volatility levels are $\sigma = 0.3, 0.4, \ldots, 1.0$. The Black-Scholes call value is computed at each volatility level and the resulting trend is analysed using tables, line plots, bar charts, incremental changes, and an approximate sensitivity ratio $\Delta C/\Delta\sigma$.

For the inferential component, the null hypothesis is

$$H_0 : \mu_{0.3} = \mu_{0.4} = \cdots = \mu_{1.0},$$

against

$$H_1 : \text{at least one mean call value differs due to volatility.}$$

The non-central F interpretation is based on the alternative distribution

$$F \sim F'(df_1, df_2, \lambda),$$

where the non-centrality parameter is

$$\lambda = \frac{\sum n_i(\mu_i - \mu)^2}{\sigma_e^2}. \tag{15}$$

This statistic measures how far the true group means are separated relative to residual variation.

4. Results and Graphical Analysis

The computed call option values are reported in Table 1. The values indicate that the call price increases with volatility for both initial stock price levels.

Table 1: Analysis of call option results

Volatility (σ)	$S_0 = 60$ BS value	$S_0 = 70$ BS value	ΔC_{60}	ΔC_{70}
0.3	39.53	49.53	–	–
0.4	39.55	49.54	0.02	0.01
0.5	39.63	49.57	0.08	0.03
0.6	39.83	49.69	0.20	0.12
0.7	40.16	49.93	0.33	0.24
0.8	40.63	50.30	0.47	0.37
0.9	41.20	50.78	0.57	0.48
1.0	41.86	51.37	0.66	0.59

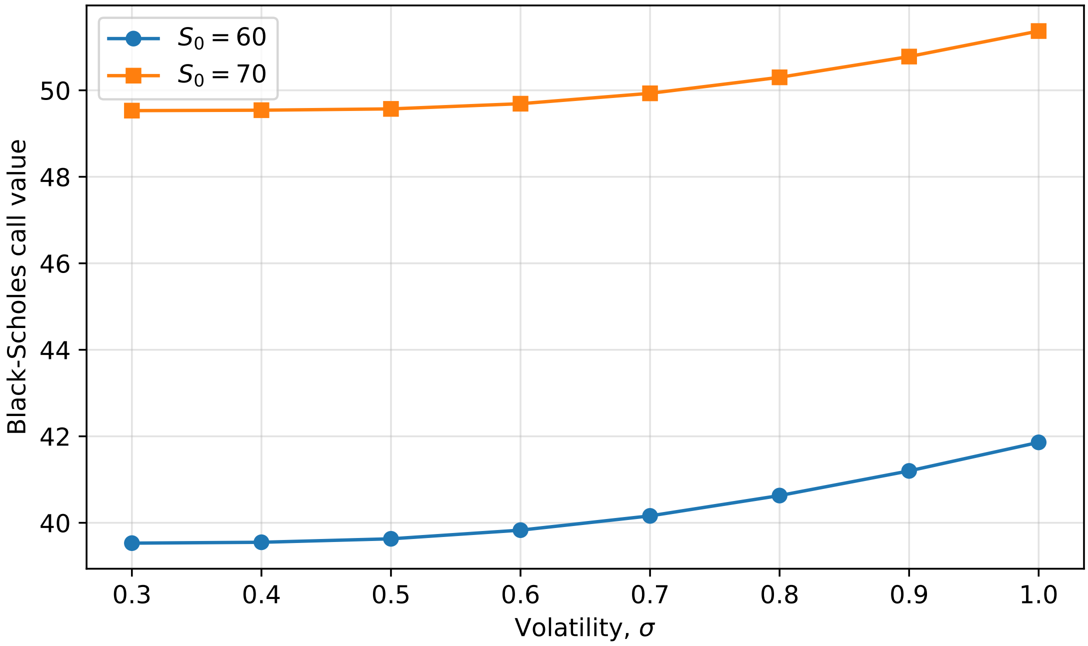

Figure 1: Black-Scholes call value against volatility for $S_0 = 60$ and $S_0 = 70$.

Figure 1 shows a monotonic increase in call option value as volatility rises. The pattern is convex, particularly for $S_0 = 60$, indicating that the marginal contribution of volatility becomes stronger at higher volatility levels.

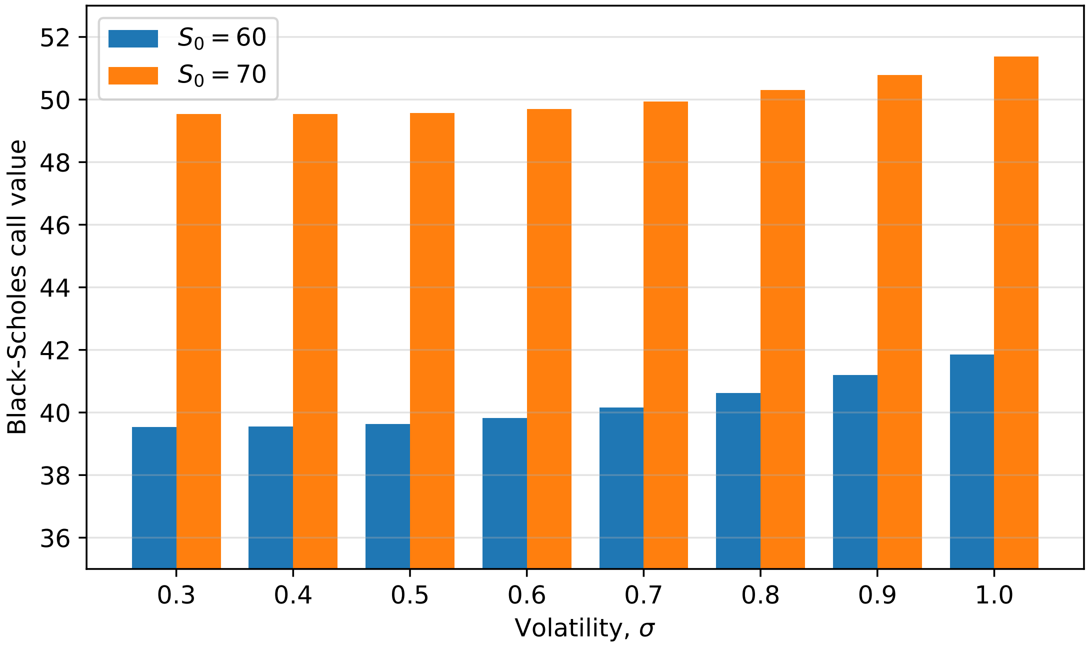

Figure 2: Grouped bar chart of Black-Scholes values across volatility levels.

Figure 2 compares the two spot-price cases at each volatility point. The $S_0 = 70$ option remains approximately \$10 above the $S_0 = 60$ option, which is close to the difference in intrinsic value. This confirms that the higher spot value shifts the price level upward while preserving the volatility-response pattern.

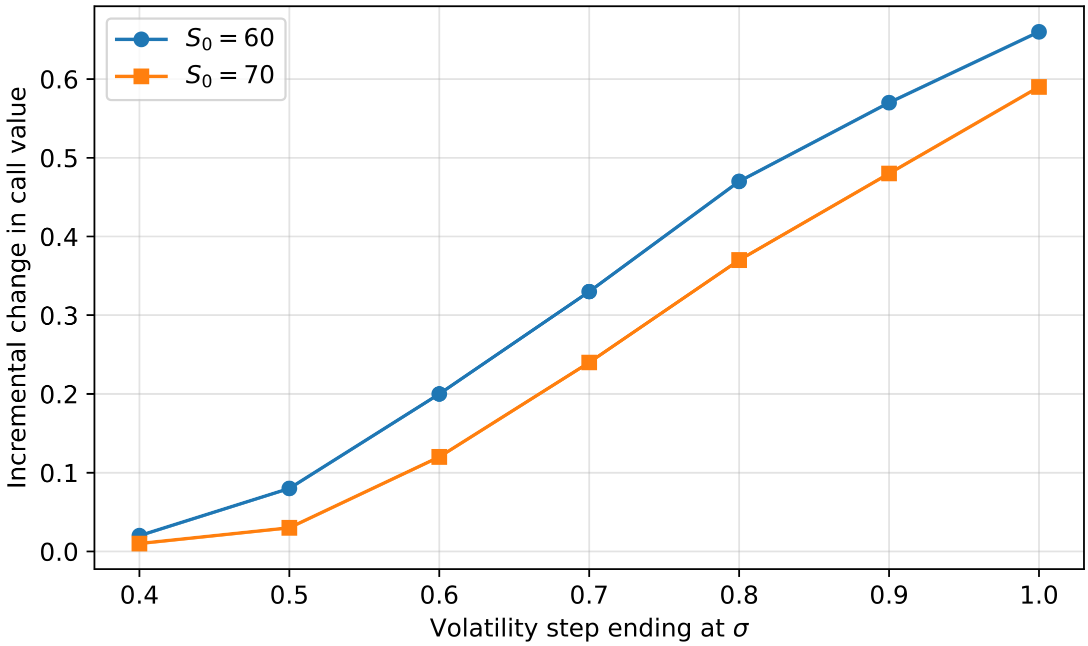

Figure 3: Incremental change in Black-Scholes call value.

Figure 3 shows that the increase in call price is not constant. The incremental change rises as volatility increases, supporting the observation that the relationship is nonlinear and accelerating.

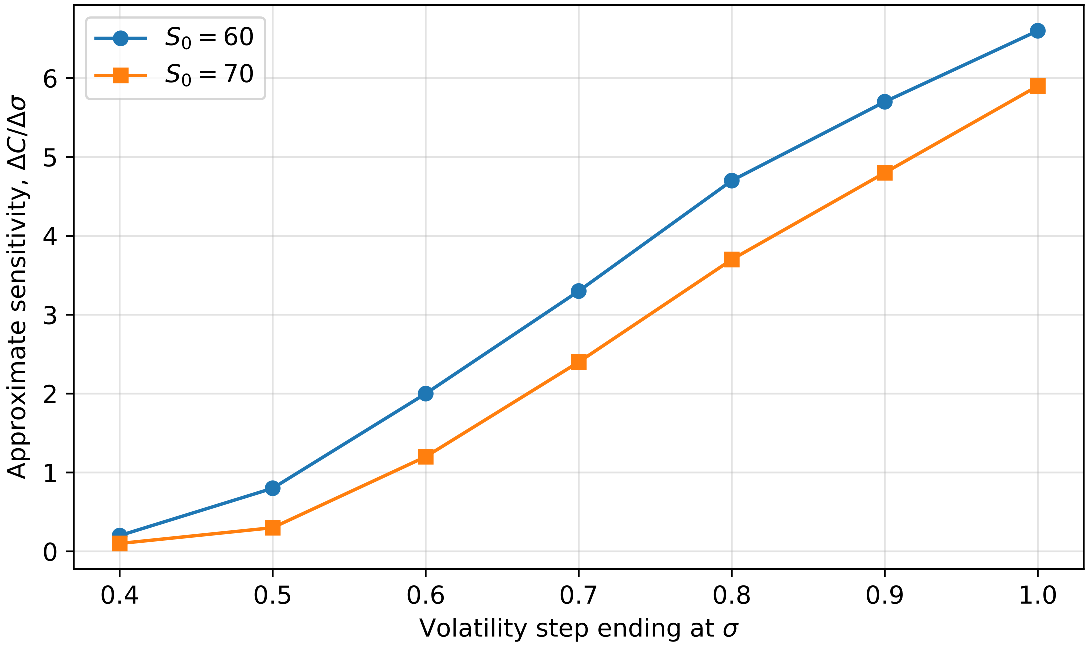

Figure 4: Approximate volatility sensitivity computed as $\Delta C/\Delta\sigma$.

The approximate sensitivity plot in Figure 4 gives the discrete vega-like response of the option price to volatility. For $S_0 = 60$, the sensitivity rises from 0.20 to 6.60 across the volatility grid. For $S_0 = 70$, it rises from 0.10 to 5.90. This shows that even deep in-the-money call options retain meaningful sensitivity to volatility.

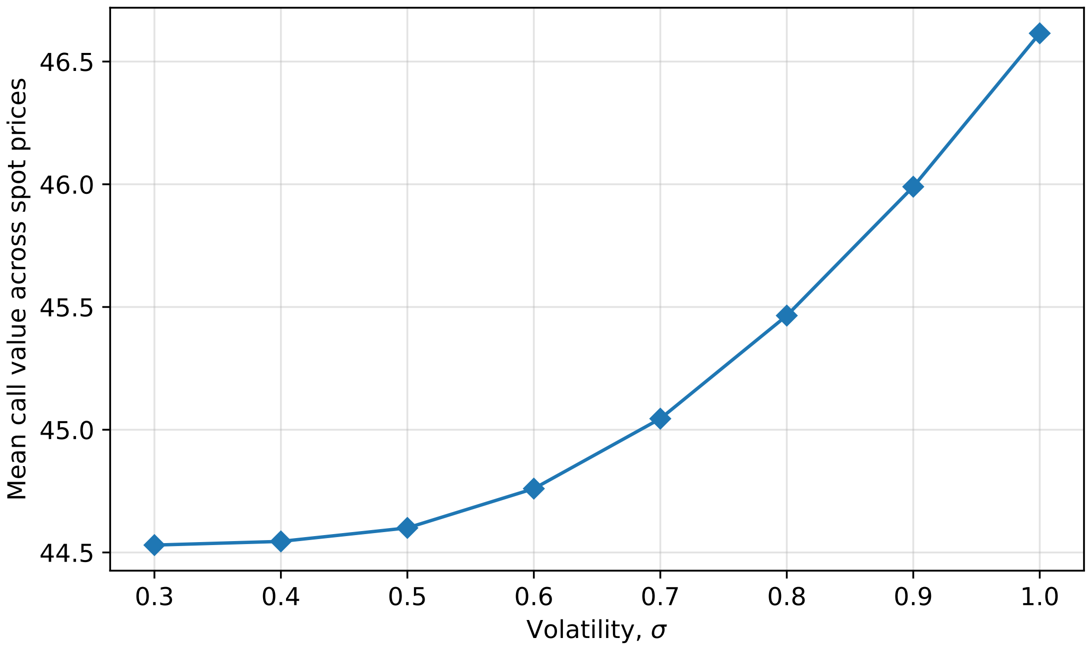

Figure 5: Mean response of call option values across both stock-price levels.

Figure 5 summarizes the average response across the two spot-price cases. The mean call value rises from 44.53 at $\sigma = 0.3$ to 46.62 at $\sigma = 1.0$, indicating a total mean increase of 2.09.

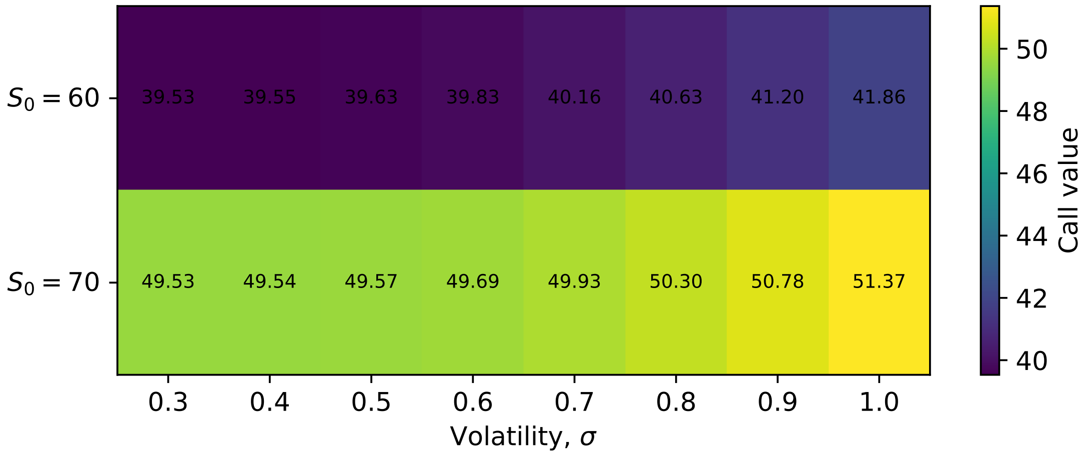

Figure 6: Heatmap of call option values by volatility and stock price.

Figure 6 provides a compact view of the price surface. Higher values occur at higher stock prices and higher volatility levels. The heatmap therefore confirms the joint effect of spot price and volatility on the European call value.

5. Statistical Analysis and Discussion

From $\sigma = 0.3$ to $\sigma = 1.0$, the call value increases by \$2.33 for $S_0 = 60$ and by \$1.84 for $S_0 = 70$. In percentage terms, this corresponds to increases of approximately 5.9% and 3.7%, respectively. The effect is nonlinear and accelerating because the incremental changes become larger as volatility increases.

The two price series differ mostly by the spot-price level. For the same volatility, the call value for $S_0 = 70$ is approximately \$10 higher than the call value for $S_0 = 60$. This is consistent with the intrinsic value contribution of the higher underlying asset price. However, the graphical evidence also shows that volatility has a clear effect on both series. A blocked regression of call value on volatility and stock-price level gives the fitted relation

$$\widehat{C} = -20.3526 + 2.9405\sigma + 0.9790S_0,$$

with $R^2 = 0.9970$. The partial F-statistic for the volatility term, after controlling for the spot-price level, is approximately $F = 81.05$. This supports rejection of the null hypothesis and indicates that volatility has a statistically meaningful effect on the computed option values.

The non-central F framework further explains this result under the alternative hypothesis. A large non-centrality parameter implies that the observed separation in mean call values across volatility levels is too large to be attributed to residual variation alone. Therefore, volatility has a significant and economically relevant effect on deep in-the-money European call option prices. This supports the need for careful volatility calibration in pricing and hedging practice, especially in high interest-rate environments.

6. Conclusion

The Black-Scholes model predicts that call option values increase with volatility even when options are deep in-the-money. For $K = 25$, moving σ from 0.3 to 1.0 raises the $S_0 = 60$ call by 5.9% and the $S_0 = 70$ call by 3.7%. The graphical analysis confirms a monotonic, nonlinear, and accelerating effect of volatility on call value. The non-central F and blocked-regression interpretation validates that this relationship is statistically meaningful and not merely a random fluctuation in the computed results. Practically, traders and risk managers pricing deep in-the-money calls in volatile and high-interest-rate markets must calibrate σ carefully, because small volatility mis-specifications may lead to material mispricing. Future work should extend the analysis to stochastic volatility models and compare Black-Scholes vega to model-free volatility measures.

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Conflict of Interest

The authors declare no conflict of interest.

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REFERENCES

References

[1] Amadi, I. U., and Umoh, B. E. (2023). Analytical method for estimation of Black-Scholes equation of option pricing. *International Journal of Mathematical Analysis and Modelling*, 5(4), 1–13.

[2] Mandah, O. C., Charles, A., and Amadi, I. U. (2025). Crank-Nicolson analysis of Black-Scholes equation and effects of covariance properties for stock market prices. *Asian Journal of Pure and Applied Mathematics*, 7(1), 424–436.

[3] Kwok, Y. K. (2008). *Mathematical Models of Financial Derivatives*. Springer, Singapore.

[4] Hull, J. C. (2012). *Options, Futures and Other Derivatives* (8th ed.). Prentice Hall, Boston.

[5] Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), 637–654.

[6] Marcelo, B., Scott, M., and Marco, S. (2014). Implied volatility and the risk-free rate of return in option markets. Preliminary draft.

[7] Babasola, O. L., Irakoze, I., and Onoja, A. A. (2008). Valuation of European options within the Black-Scholes framework using the Hermite polynomial. *Journal of Scientific and Engineering Research*, 5(1), 213–221.

[8] Shin, B., and Kim, H. (2016). The solution of Black-Scholes terminal value problem by means of Laplace transform. *Global Journal of Pure and Applied Mathematics*, 12, 4153–4158.

[9] Rodrigo, M. R., and Mamon, R. S. (2006). An alternative to solving the Black-Scholes equation with time-varying parameters. *Applied Mathematics*, 19, 398–402.

[10] Osu, B. O., Okoroafor, A. C., and Olunkwa, C. (2009). Stability analysis of stochastic model of stock market price. *African Journal of Mathematics and Computer Science Research*, 2(6), 98–103.

[11] Osu, B. O. (2010). A stochastic model of the variation of the capital market price. *International Journal of Trade, Economics and Finance*, 1(3), 297–302.

[12] Nwobi, F. N., Annorzie, M. N., and Amadi, I. U. (2019). Crank-Nicolson finite difference method in valuation of options. *Communications in Mathematical Finance*, 8(1), 93–122.

[13] Bachelier, L. (1964). *Théorie de la Spéculation*. Gauthier-Villars, Paris.

[14] Westergren, B., and Råde, L. (2003). *Mathematics Handbook for Science and Engineering*. Studentlitteratur AB.

[15] Hull, J. C. (2003). *Options, Futures and Other Derivatives* (5th ed.). Prentice Hall International, London.

[16] Higham, D. J. (2004). *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation*. Cambridge University Press.

[17] Wilmott, P., Howison, S., and Dewynne, J. (2008). *The Mathematics of Financial Derivatives*. Cambridge University Press, New York.

