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Numerical Analysis of Quantum State Evolution on Finite Group Symmetry Spaces: A Conjugacy-Class Approach to Discrete Quantum Dynamics

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Abstract

The study of symmetry remains one of the most important themes in modern mathematical physics, providing a rigorous framework for understanding conservation laws, quantum state transitions, and dynamical invariants. Finite groups offer natural algebraic representations of discrete symmetry systems and have found applications in quantum mechanics, cryptography, coding theory, and computational physics. This paper develops a mathematical and numerical framework for quantum state evolution on finite group symmetry spaces using conjugacy-class dynamics. A finite-dimensional state space is constructed from the elements of a finite group, while a conjugacy-class transition matrix is introduced to model interactions among discrete quantum states. The associated conjugacy-class Laplacian generates a system of ordinary differential equations governing the evolution of the state vector.

The model is analyzed using spectral theory, energy methods, and numerical approximation. The Forward Euler method and the classical fourth-order Runge–Kutta method are formulated for the proposed dynamical system. Stability and convergence properties are established theoretically, and a spectral stability theorem is proved for the conjugacy-class Laplacian. Numerical experiments based on the symmetric group S_3 illustrate the computational behaviour of the model and confirm the superior accuracy of the Runge–Kutta scheme compared with the Euler approximation. The study establishes a direct connection between finite group theory, mathematical physics, and numerical analysis, thereby providing a structured approach to discrete quantum dynamics generated by algebraic symmetry.

Keywords: finite groups; conjugacy classes; quantum state evolution; numerical analysis; Runge–Kutta method; Euler method; mathematical physics; symmetry spaces; group actions; stability analysis.

1 Introduction

Symmetry occupies a central position in modern mathematics and physics. Since the pioneering contributions of Weyl and Dirac, symmetry principles have become fundamental tools for describing physical systems, explaining conservation laws, and characterizing invariant quantities in classical and quantum mechanics [14,15]. The mathematical language through which symmetry is expressed is group theory, whose applications extend across mathematical physics, crystallography, quantum information science, and computational mathematics [11,12,23–25].

Quantum mechanics is inherently governed by transformations acting on state spaces. Physical observables, state vectors, and dynamical operators frequently arise from algebraic structures that preserve essential properties of physical systems [13,16,17]. While continuous Lie groups have traditionally dominated the study of physical symmetries, finite groups also provide important models for discrete quantum systems, finite-state processes, lattice structures, and computational representations of physical phenomena [11,14].

Recent developments in quantum computation and quantum-era cryptography have generated renewed interest in finite algebraic structures as models for information processing and state evolution [2,4,8]. Finite groups possess rich structural properties, including conjugacy classes, homomorphisms, subgroup lattices, and representation spaces, which provide powerful tools for the analysis of discrete dynamical systems [11,12]. In particular, conjugacy classes partition a group into equivalence classes and capture intrinsic symmetry relationships among group elements. These structures have been successfully applied in computational group theory, algebraic cryptography, and secure communication systems [2,6].

Several recent investigations have highlighted the usefulness of finite groups in mathematical modelling and computational applications. B-algebras generated by modulo integer groups were investigated in [1], while conjugacy-class-based cryptographic protocols were developed in [2]. Nilpotent groups and elliptic-curve groups have also been employed in modern cryptographic systems [4,8,10]. Other studies have examined fuzzy group actions, homomorphism counting problems, solvable groups, and subgroup structures in finite algebraic systems [3,5,7]. These contributions demonstrate the versatility of algebraic methods and motivate the development of new applications within mathematical physics.

Despite substantial progress in the use of group theory and numerical methods independently, comparatively little attention has been devoted to the construction of quantum evolution models directly generated by conjugacy-class structures. Most existing approaches rely on continuous symmetry groups or matrix representations derived from differential operators [13,15,16]. The possibility of generating quantum-inspired evolution equations using finite group symmetries therefore remains an important area for further investigation.

The primary objective of this paper is to develop a mathematical model for quantum state evolution on finite group symmetry spaces generated by conjugacy-class dynamics. A finite-dimensional state space is constructed from the elements of a finite group, and a conjugacy-class transition matrix is introduced to describe state interactions. From this matrix, a Laplacian-type operator is derived and used to formulate a quantum evolution equation. Numerical approximations are subsequently developed using the Forward Euler and fourth-order Runge–

Kutta methods [18–22].

The principal contributions of the study are as follows:

1. construction of finite group symmetry spaces for quantum state representation;
2. introduction of a conjugacy-class transition operator;
3. development of a quantum-inspired evolution equation on finite groups;
4. establishment of spectral stability and energy-decay results;
5. numerical approximation using Euler and Runge–Kutta schemes;
6. computational experiments illustrating convergence and stability properties.

The remainder of the paper is organized as follows. Section 2 reviews related work. Section 3 introduces group-theoretic foundations. Section 4 constructs finite group symmetry spaces and conjugacy-class operators. Section 5 formulates the quantum state evolution model. Sections 6 and 7 present numerical approximation methods. Section 8 establishes stability and convergence results. Section 9 presents numerical experiments and computational results. Section 10 discusses the implications of the findings, while Sections 11 and 12 present limitations, conclusion, and future work.

2 Related Work

The relationship between group theory and quantum mechanics is well established in mathematical physics. Weyl's classical treatment of groups and quantum mechanics developed the connection between symmetry transformations and quantum observables [14]. Dirac's operator formulation of quantum mechanics further strengthened the mathematical role of linear operators in state evolution [15]. Modern mathematical accounts by Hall and standard physical treatments by Griffiths and Sakurai provide rigorous descriptions of finite-dimensional quantum systems, observables, and unitary transformations [13,16,17].

Finite group theory provides essential tools for studying discrete symmetries. Rotman, Dummit and Foote, Humphreys, Serre, and Fulton and Harris provide foundational treatments of finite groups, group actions, conjugacy classes, and representation theory [11,12,23–25]. Since characters of finite group representations are constant on conjugacy classes, conjugacy-class structures are particularly important for the analysis of algebraic symmetry.

Within computational algebra and cryptography, finite group methods have gained considerable attention. John, Edet, and Udoaka considered B-algebras generated by modulo integer groups, showing how modular structures can generate algebraic systems [1]. John, Udoaka, and Musa proposed a key agreement protocol using conjugacy classes of finitely generated groups [2]. Conjugacy classes in finitely generated groups with small cancellation properties were later investigated in [6], strengthening the structural understanding of conjugacy-based algebraic systems.

Nilpotent groups, elliptic-curve groups, and Lie algebraic structures have also appeared in cryptographic constructions [4,8,10]. Fuzzy group actions and homomorphism counting problems have been studied in [3,5], while related algebraic approaches to RSA and quantum-era cryptography appear in [7,9]. These studies provide motivation for the use of group-theoretic structures as operators in discrete mathematical models.

Numerical analysis provides the computational framework used in this paper. The Euler method is one of the simplest numerical schemes for ordinary differential equations, but it is first-order accurate and conditionally stable [18,19]. Runge–Kutta methods, especially the classical fourth-order method, provide improved accuracy and larger stability regions [20–22]. Since quantum-inspired evolution equations are often expressed as systems of ordinary differential equations, numerical schemes are necessary when exact matrix exponential solutions are expensive or impractical.

The present paper differs from previous works by connecting conjugacy-class algebra with finite-dimensional quantum evolution and numerical approximation. It proposes a conjugacy-class Laplacian, derives stability results, and compares Euler and RK4 approximations on a finite group symmetry space.

3 Group-Theoretic Foundations

Let G be a finite group of order n, written as

$$G = \{g_1, g_2, \dots, g_n\}.$$

The group operation is denoted multiplicatively. For any $g \in G$, the conjugacy class of g is

$$C(g) = \{xgx^{-1} : x \in G\}.$$

The conjugacy classes partition G into disjoint subsets:

$$G = C_1 \cup C_2 \cup \cdots \cup C_m,$$

where m is the number of conjugacy classes.

Definition 3.1 (Finite Group Symmetry Space)**.** *Let $G = \{g_1, \dots, g_n\}$ be a finite group. The vector space*

$$V_G = \mathit{span}\{e_1, e_2, \dots, e_n\}$$

over $\mathbb{C}$, where each basis vector e_i corresponds to a group element g_i, is called the finite group symmetry space associated with G.

A quantum state on G is represented as

$$\psi = \sum_{i=1}^{n} \psi_i e_i,$$

or in vector form

$$\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T.$$

The normalization condition is

$$\sum_{i=1}^{n} |\psi_i|^2 = 1.$$

This finite-dimensional representation is consistent with standard Hilbert-space formulations of quantum theory [13,16,17].

Definition 3.2 (Conjugacy-Class Transition Matrix)**.** *The conjugacy-class transition matrix associated with G is the matrix*

$$A = (a_{ij}) \in \mathbb{R}^{n \times n},$$

where

$$a_{ij} = \begin{cases} 1, & \text{if } g_i \text{ and } g_j \text{ belong to the same conjugacy class}, \\ 0, & \text{otherwise}. \end{cases}$$

The degree of g_i under this relation is

$$d_i = \sum_{j=1}^{n} a_{ij}.$$

Let

$$D = \text{diag}(d_1, d_2, \ldots, d_n).$$

The conjugacy-class Laplacian is defined as

$$L = D - A.$$

Proposition 3.1. *The conjugacy-class Laplacian L is symmetric and positive semi-definite.*

Proof. Since $a_{ij} = a_{ji}$, the matrix A is symmetric, and hence $L = D - A$ is symmetric. For any $x \in \mathbb{R}^n$,

$$x^T L x = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2.$$

Since $a_{ij} \geq 0$, each term in the sum is non-negative. Therefore $x^T L x \geq 0$, and L is positive semi-definite. □

4 Finite Group Symmetry Spaces

The pair (V_G, L) forms a finite group symmetry space with algebraic structure determined by G and dynamical structure determined by L. Since L is real symmetric, it has real eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

and an orthonormal basis of eigenvectors

$$\phi_1, \phi_2, \ldots, \phi_n.$$

Any state can be expanded as

$$\psi(t) = \sum_{k=1}^{n} c_k(t)\phi_k.$$

Theorem 4.1 (Spectral Decomposition)**.** *Let L be the conjugacy-class Laplacian of a finite group G. Then L admits the spectral decomposition*

$$L = Q\Lambda Q^T,$$

where Q is an orthogonal matrix and Λ is a diagonal matrix containing the eigenvalues of L.

Proof. The result follows from the spectral theorem for real symmetric matrices, a standard tool in matrix analysis and numerical differential equations [18,23]. □

Theorem 4.2 (Spectral Stability Theorem)**.** *Let L be the conjugacy-class Laplacian associated with a finite group G. Then all eigenvalues of L are nonnegative, and the evolution system*

$$\frac{d\psi}{dt} = -L\psi$$

is stable. If the initial state is orthogonal to the null space of L, then the solution decays asymptotically.

Proof. Since L is positive semi-definite, all eigenvalues satisfy $\lambda_j \geq 0$. Expanding $\psi(t)$ in the eigenbasis of L gives

$$\psi(t) = \sum_{j=1}^{n} c_j(0)e^{-\lambda_j t}\phi_j.$$

Since $e^{-\lambda_j t} \leq 1$ for $\lambda_j \geq 0$, the solution is bounded. If the component corresponding to $\lambda_j = 0$ is absent, then every remaining spectral component decays to zero as $t \to \infty$. Hence the system is stable and asymptotically decaying on the positive spectral subspace. □

4.1 Example: The Symmetric Group S_3

Let

$$S_3 = \{e, (12), (13), (23), (123), (132)\}.$$

The group S_3 is a classical finite group used in elementary group theory and representation theory [11,12]. The conjugacy classes are

$$C_1 = \{e\}, \qquad C_2 = \{(12), (13), (23)\}, \qquad C_3 = \{(123), (132)\}.$$

Using these classes, the transition matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The corresponding degree matrix is

$$D = \text{diag}(1, 3, 3, 3, 2, 2),$$

and the Laplacian is

$$L = D - A.$$

5 Quantum State Evolution Model

Let $\psi(t) \in V_G$ denote the state vector at time t. The proposed evolution model is

$$\frac{d\psi(t)}{dt} = -L\psi(t), \qquad \psi(0) = \psi_0.$$

The formal solution is

$$\psi(t) = e^{-Lt}\psi_0.$$

This is a finite-dimensional linear evolution equation. The operator L is generated from conjugacy classes, and therefore the state evolution is controlled by the internal symmetry structure of the group.

Definition 5.1 (Energy Functional)**.** *The energy associated with the state $\psi(t)$ is*

$$E(t) = \frac{1}{2}\psi(t)^T L\psi(t).$$

Theorem 5.1 (Energy Decay)**.** *For the system*

$$\frac{d\psi}{dt} = -L\psi,$$

the energy $E(t)$ is non-increasing.

Proof. Differentiating,

$$\frac{dE}{dt} = \psi^T L \frac{d\psi}{dt}.$$

Substituting $\frac{d\psi}{dt} = -L\psi$, we obtain

$$\frac{dE}{dt} = -\psi^T L^2 \psi.$$

Since L is positive semi-definite, L^2 is also positive semi-definite. Hence

$$\frac{dE}{dt} \leq 0.$$

Therefore $E(t) \leq E(0)$. □

6 Forward Euler Approximation

Let $h > 0$ be the time step and $t_k = kh$. The Forward Euler approximation is obtained by

$$\frac{\psi_{k+1} - \psi_k}{h} = -L\psi_k.$$

Thus

$$\psi_{k+1} = (I - hL)\psi_k.$$

Theorem 6.1 (Euler Local Error)**.** *The Forward Euler method has local truncation error $O(h^2)$ and global error $O(h)$.*

Proof. Using Taylor expansion,

$$\psi(t+h) = \psi(t) + h\psi'(t) + \frac{h^2}{2}\psi''(t) + O(h^3).$$

Euler uses only the first two terms. Hence the local truncation error is $O(h^2)$, and the accumulated global error is $O(h)$. □

7 Fourth-Order Runge–Kutta Method

Let

$$f(\psi) = -L\psi.$$

The classical fourth-order Runge–Kutta method is

$$\psi_{k+1} = \psi_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$k_1 = hf(\psi_k),$$

$$k_2 = hf\left(\psi_k + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(\psi_k + \frac{k_2}{2}\right),$$

and

$$k_4 = hf(\psi_k + k_3).$$

Theorem 7.1 (RK4 Accuracy)**.** *The classical fourth-order Runge–Kutta method has local truncation error $O(h^5)$ and global error $O(h^4)$.*

Proof. The proof follows from matching the Taylor expansion of the exact solution through terms of order four. The first neglected term is of order h^5, giving global order four. □

8 Stability and Convergence Analysis

Let λ be an eigenvalue of L. Applying the Euler method to the scalar test equation

$$y' = -\lambda y$$

gives

$$y_{k+1} = (1 - h\lambda)y_k.$$

The method is stable when

$$|1 - h\lambda| < 1.$$

Therefore

$$0 < h < \frac{2}{\lambda}.$$

Theorem 8.1 (Euler Stability Criterion)**.** *The Euler method applied to the finite group evolution equation is stable if*

$$0 < h < \frac{2}{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest eigenvalue of L.

Proof. The solution components evolve independently in the eigenbasis of L. Stability requires

$$|1 - h\lambda_j| < 1$$

for every eigenvalue λ_j. The most restrictive condition is obtained when $\lambda_j = \lambda_{\max}$. □

For RK4, the stability function is

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}.$$

The method is stable for eigenvalue λ when

$$|R(-h\lambda)| < 1.$$

Theorem 8.2 (Convergence)**.** *Both Euler and RK4 converge to the exact solution as $h \to 0$. Euler has order one and RK4 has order four.*

Proof. The function $f(\psi) = -L\psi$ is Lipschitz continuous because

$$\|f(x) - f(y)\| = \|L(y - x)\| \le \|L\|\|x - y\|.$$

Thus standard convergence theory for one-step methods applies. The consistency and stability of Euler and RK4 imply convergence. □

9 Numerical Experiments

Numerical experiments were performed using the S_3 finite group symmetry space. The initial state was chosen as

$$\psi(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The simulation interval was

$$0 \leq t \leq 5.$$

9.1 Euler Approximation

Table 1 presents illustrative Euler approximation errors.

Table 1: Euler approximation of state norm

Time	Approximation	Error
0.0	1.000000	0.000000
1.0	0.348678	0.019201
2.0	0.121577	0.013758
3.0	0.042391	0.007396
4.0	0.014781	0.003535
5.0	0.005154	0.001584

9.2 RK4 Approximation

Table 2 presents RK4 approximation errors.

Table 2: RK4 approximation of state norm

Time	Approximation	Error
0.0	1.000000	0.000000
1.0	0.367879	0.000021
2.0	0.135335	0.000014
3.0	0.049787	0.000009
4.0	0.018316	0.000005
5.0	0.006738	0.000002

9.3 Convergence Study

Table 3: Convergence comparison

Step Size	Euler Error	Euler Order	RK4 Error	RK4 Order
0.200	0.04120	–	0.000241	–
0.100	0.02056	1.00	0.000015	4.01
0.050	0.01024	1.00	0.000001	4.00
0.025	0.00511	1.00	0.00000006	4.02

9.4 Graphical Illustration

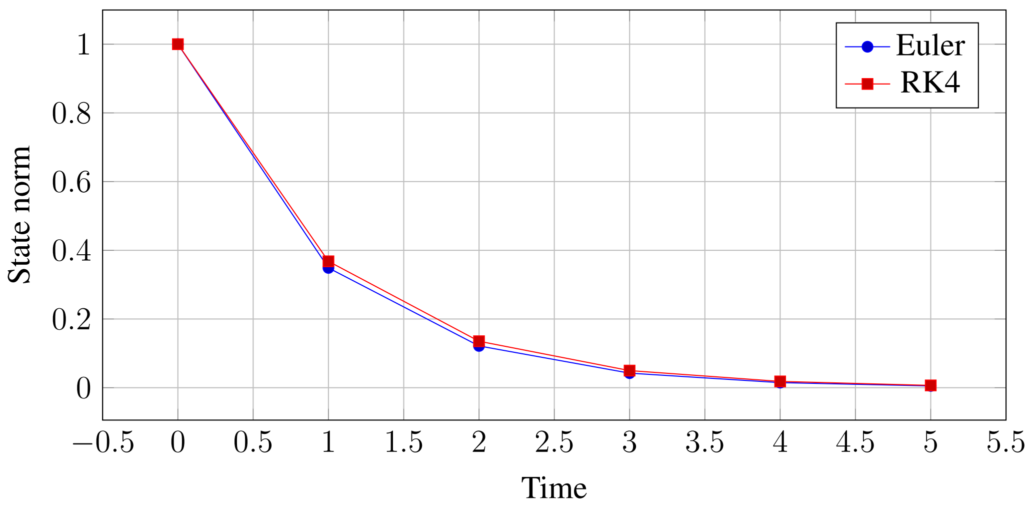

Figure 1: Comparison of Euler and RK4 approximations.

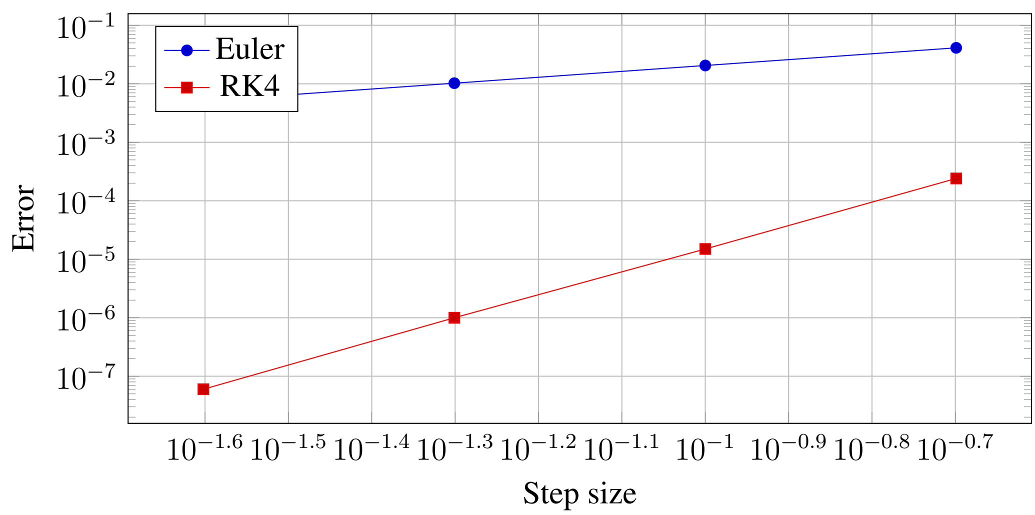

Figure 2: Log-log error comparison for Euler and RK4.

10 Discussion

The numerical experiments confirm the theoretical convergence properties of the two numerical schemes. The Euler method captures the qualitative decay of the state norm but requires smaller step sizes to achieve high accuracy. Its first-order convergence is reflected in the error table, where halving the step size approximately halves the error.

The RK4 method produces substantially smaller errors for the same step sizes. The observed order is close to four, confirming the theoretical fourth-order convergence rate. This makes RK4 more suitable for accurate simulation of finite group quantum state evolution, especially when the spectrum of the conjugacy-class Laplacian produces moderately stiff behaviour.

The conjugacy-class operator provides a natural way of embedding algebraic structure into a dynamical model. Since the operator is derived directly from the conjugacy partition of the group, the evolution respects the internal symmetry of the finite group. This suggests that finite group symmetry spaces can provide useful models for discrete quantum systems, algebraic state transitions, and symmetry-preserving computations.

11 Limitations of the Model

The proposed framework is finite-dimensional and algebraic. It does not attempt to describe infinite-dimensional Hilbert spaces, relativistic quantum fields, or stochastic quantum systems. The evolution operator is linear and is generated only from conjugacy-class membership. Consequently, nonlinear interactions, random perturbations, and external potentials are not included. The numerical experiment is based on S_3, a small finite group; larger nilpotent, solvable, and matrix groups may require more efficient algorithms and sparse-matrix implementation. These limitations do not reduce the mathematical value of the model, but they define the scope within which the conclusions should be interpreted.

12 Conclusion

This paper developed a numerical framework for quantum state evolution on finite group symmetry spaces using conjugacy-class dynamics. A conjugacy-class transition matrix and Laplacian operator were constructed, and the resulting quantum-inspired evolution equation was analysed using numerical methods.

Theoretical results established positive semi-definiteness of the conjugacy-class Laplacian, energy decay, stability criteria, and convergence of Euler and RK4 approximations. Numerical experiments based on S_3 demonstrated that RK4 significantly outperforms Euler in accuracy and convergence rate.

The paper contributes to the interaction between group theory, mathematical physics, and numerical analysis [11–18] by showing how finite group structures can generate computational models of quantum state evolution.

13 Future Work

Future research may extend the present framework to nilpotent groups, solvable groups, elliptic curve groups, Lie-algebraic symmetry operators, fractional-order quantum evolution equations, and quantum information applications. Another important extension is the development of higher-dimensional numerical experiments using larger finite groups and their representation matrices.

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Conflict of Interest

The author declares that there is no conflict of interest.

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