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On the Identity of Implicit Deformation Geometry Generated by a Nonlinear Balance Law

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Abstract

We introduce and study an implicit deformation geometry generated by the nonlinear balance law $R^p + qR = e^v$, where $p \in (0,1]$, $q \geq 0$, and $R = R(v;p,q)$ is defined implicitly. The associated compactified deformator is $\phi(v;p,q) = \frac{R(v;p,q)}{1+R(v;p,q)}$, which induces a localization geometry on $(0,1)$. We establish the existence and uniqueness of the implicit branch, smooth dependence on transport and deformation parameters, transport sensitivity, biparametric deformation geometry, localization amplification, damping behavior, and asymptotic concentration. Explicit formulas are derived for ϕ_v, ϕ_p, and ϕ_q, revealing the parameter-sensitive structure of the induced geometry. The resulting framework provides a non-Jacobian localization geometry generated intrinsically through nonlinear equilibrium rather than explicit coordinate transport.

Keywords: implicit deformation geometry; nonlinear balance law; compactified deformator; localization geometry; transport sensitivity; non-Jacobian geometry

1 Introduction

Classical transport geometry is predominantly constructed via explicit coordinate transformations, differentiable flows, and prescribed Jacobian structures. These methodologies form the foundation for expansive domains within functional analysis, operator theory, and geometric measure theory (cf. Lang [1], Rudin [6]). In such configurations, a reference measure or spatial domain is systematically distorted by an explicitly defined velocity field, meaning the underlying geometric properties are determined directly by the derivative properties of the mapping itself.

However, various complex physical and analytical systems exhibit localized phenomena that are not naturally captured by explicit transport maps. In systems governed by multi-scale interactions, non-local mechanics, or competing constitutive relations, localization often emerges as an asymptotic equilibrium state rather than an externally driven kinematical consequence. To model such behaviors without relying on manually engineered coordinate profiles, it is necessary to establish an alternate framework where geometric structure is generated implicitly through local algebraic or functional constraints.

The objective of this work is to introduce a non-Jacobian geometric framework wherein localization geometry is generated implicitly through a nonlinear balance law. The foundational generating equation is given by:

$$R^p + qR = e^v \tag{1}$$

where $v \in \mathbb{R}$ represents the underlying transport variable, $p \in (0, 1]$ is a nonlinear deformation parameter, $q \geq 0$ is a structural transport damping parameter, and $R = R(v; p, q) > 0$ is the unique positive scalar field defined implicitly by the equilibrium. To map this unbounded implicit branch onto a bounded geometric domain, we introduce the compactified implicit deformator $\phi : \mathbb{R} \to (0, 1)$, defined by:

$$\phi(v; p, q) = \frac{R(v; p, q)}{1 + R(v; p, q)} \tag{2}$$

This definition establishes a mathematically coherent convention used uniformly throughout this paper: as $v \to -\infty$, the implicit solution vanishes ($R \to 0^+$), causing the deformator to approach its lower boundary ($\phi \to 0^+$); conversely, as $v \to +\infty$, the implicit solution diverges ($R \to \infty$), driving the deformator toward unity ($\phi \to 1^-$).

The core conceptual novelty of this framework lies in the principle that **localization geometry can emerge intrinsically from an implicit nonlinear balance law rather than from an explicitly prescribed coordinate transformation or Jacobian transport mechanism**. Within this structural paradigm, the classical causal chain of geometric transport is inverted: *nonlinear equilibrium generates transport*, *transport generates localization*, and *localization generates deformation geometry*.

By avoiding an explicitly prescribed Jacobian mapping, the geometry becomes highly responsive to variations in its internal parameters. The exponent p acts as a nonlinear deformation amplifier that modifies micro-scale spatial concentration, while the coefficient q serves as a macro-scale transport damping parameter that smooths localized gradients. This framework relates conceptually to developments in implicit function geometry, generalized logistic growth models, and non-linear equilibrium-generated manifolds found in modern mathematical physics.

The remainder of this paper is structured as follows. Section 2 establishes the analytical foundation of the framework, proving the existence, uniqueness, and global smoothness of the implicit branch $R(v; p, q)$ via the Implicit Function Theorem. Section 3 develops the intrinsic differential geometry of the deformator, deriving the exact transport sensitivity relations. Section 4 explores the biparametric deformation geometry, evaluating how the interaction between p and q induces localized structural transitions. Section 5 evaluates the left and right localization asymptotics to confirm spatial concentration behaviors. Section 6 formalizes the canonical localization regimes, and Section 7 frames the structural utility of the framework alongside future directions in non-Jacobian operator theory.

2 The Implicit Deformation Framework

To formalize the structural origin of the geometry, let $F : (0, \infty) \times \mathbb{R} \times (0, 1] \times [0, \infty) \to \mathbb{R}$ be the smooth generating functional defined by:

$$F(R, v; p, q) = R^p + qR - e^v \tag{3}$$

Definition 2.1 (The John Implicit Biparametric Deformator System)**.** Let $\mathbf{P} = (0, 1] \times [0, \infty)$ denote the parameter space. Following the functional foundations established by Eno John [16] for deformation geometry and localized operators, the *John Implicit Biparametric Deformator System* is defined as the coupled geometric system $(\phi_{(p,q)}, R)$ mapping the real coordinate track $v \in \mathbb{R}$ onto the bounded interval $(0, 1)$ via the simultaneous relation:

$$F(R, v; p, q) = 0, \qquad \phi_{(p,q)}(v) = \frac{R(v; p, q)}{1 + R(v; p, q)} \tag{4}$$

The operator $\phi_{(p,q)}$ acts as a geometric compactification mechanism, compactly mapping the unbounded, implicitly defined manifold branch $R \in (0, \infty)$ onto the bounded spatial domain $(0, 1)$.

Definition 2.2 (Localization Geometry)**.** The metric and structural properties induced by the compactification map ϕ on the open interval $(0, 1)$ define the *implicit localization geometry* generated by the underlying nonlinear balance law.

Theorem 2.3 (Global Existence, Uniqueness, and Regularity)**.** *For any fixed parameter pair $(p, q) \in (0, 1] \times [0, \infty)$ and for every configuration variable $v \in \mathbb{R}$, there exists a unique, strictly positive solution $R(v; p, q) \in (0, \infty)$ satisfying $F(R, v; p, q) = 0$. Furthermore, the assignment $v \mapsto R(v; p, q)$ defines a global smooth branch that delineates a one-dimensional implicit manifold.*

Proof. Fix the parameters p and q, and evaluate the coordinate mapping $G(R) = R^p + qR$ over the domain $R \in (0, \infty)$. Differentiating with respect to R yields:

$$\frac{\partial G}{\partial R} = pR^{p-1} + q$$

Given that $p \in (0, 1]$, $R > 0$, and $q \geq 0$, the derivative is strictly positive ($\frac{\partial G}{\partial R} > 0$) across the entire positive real half-line. Consequently, G is strictly monotonic and injective. Examining the boundary limits of this generator reveals:

$$\lim_{R \to 0^+} G(R) = 0 \quad \text{and} \quad \lim_{R \to \infty} G(R) = \infty$$

By the Intermediate Value Theorem, G is a continuous bijection from $(0, \infty)$ to $(0, \infty)$. Because the driving transport term satisfies $e^v > 0$ for all real v, a unique, globally defined positive solution branch exists and can be represented as $R(v; p, q) = G^{-1}(e^v)$.

To establish the regular manifold structure of this branch, we note that F is smoothly differentiable (C^∞) on its structural domain. The partial derivative of the generating functional with respect to the state coordinate R is:

$$\frac{\partial F}{\partial R} = pR^{p-1} + q$$

Because $R > 0$, this derivative never vanishes on the admissible state domain. By the Implicit Function Theorem, the local solution branch inherits the global smooth $C^\infty(\mathbb{R})$ regularity of the functional container, thereby ensuring global continuation of the smooth implicit branch. □

3 Differential Geometry of the Deformator

With the foundational framework structurally sound, we derive the exact differential calculus of the compactified space. This section defines the localized response profiles relative to the coordinate transport variable v.

Theorem 3.1 (Transport Derivative of the Implicit Manifold)**.** *The spatial derivative of the implicit branch $R(v; p, q)$ with respect to the underlying configuration variable v is given by:*

$$R_v = \frac{e^v}{pR^{p-1} + q} \tag{5}$$

Proof. Since $R(v; p, q)$ satisfies the global equilibrium constraint $F(R(v), v) = 0$, we differentiate the identity directly with respect to v via the total derivative:

$$\frac{\partial F}{\partial R} R_v + \frac{\partial F}{\partial v} = 0$$

Substituting the explicit partial derivatives $\frac{\partial F}{\partial R} = pR^{p-1} + q$ and $\frac{\partial F}{\partial v} = -e^v$ yields:

$$(pR^{p-1} + q)R_v - e^v = 0$$

Because the domain requires $R > 0$, the algebraic factor $(pR^{p-1} + q)$ never vanishes. Isolating R_v yields the formula. □

Theorem 3.2 (Transport Sensitivity of the Deformator)**.** *The compactified implicit deformator $\phi(v; p, q)$ satisfies the following exact gradient identity:*

$$\phi_v = \frac{e^v}{(1 + R)^2(pR^{p-1} + q)} \tag{6}$$

Proof. By applying the chain rule to the compactification mapping $\phi = \frac{R}{1+R}$, we determine the metric scaling factor with respect to the implicit coordinate R:

$$\phi_R = \frac{\partial}{\partial R}\left(\frac{R}{1 + R}\right) = \frac{1}{(1 + R)^2}$$

Compounding this with the intrinsic transport derivative via the chain rule yields:

$$\phi_v = \phi_R R_v = \frac{1}{(1+R)^2}\left(\frac{e^v}{pR^{p-1}+q}\right)$$

Gathering terms establishes the final sensitivity identity. □

Corollary 3.3 (Orientation Preservation and Asymptotic Attenuation)**.** *The compactification map preserves the orientation of the underlying transport variable across all coordinate states. That is:*

$$\phi_v > 0 \quad \forall v \in \mathbb{R} \tag{7}$$

Furthermore, the transport sensitivity profile exhibits asymptotic decay at both spatial extremes:

$$\lim_{v\to-\infty} \phi_v = 0 \quad \textit{and} \quad \lim_{v\to\infty} \phi_v = 0 \tag{8}$$

Proof. Since $e^v > 0$, $(1+R)^2 > 0$, and $(pR^{p-1}+q) > 0$ for all valid parameters and coordinate states, their quotient is strictly positive, proving $\phi_v > 0$.

To evaluate the left boundary limit as $v \to -\infty$, we note that $e^v \to 0^+$, which forces $R \to 0^+$. We partition the analysis into two distinct parametric regimes:

- **For** $0 < p < 1$**:** The term $pR^{p-1} = \frac{p}{R^{1-p}} \to \infty$ as $R \to 0^+$. Thus, the denominator component satisfies $(1+R)^2(pR^{p-1}+q) \to \infty$. The numerator simultaneously vanishes ($e^v \to 0$), forcing the quotient to converge to zero.
- **For** $p = 1$**:** The term $pR^{p-1}+q$ stabilizes to the constant $1+q$. As $v \to -\infty$, the derivative scales as $\phi_v \sim \frac{e^v}{1+q} \to 0$.

To evaluate the right boundary limit as $v \to \infty$, we note that $R \to \infty$. Assuming $q > 0$, the implicit branch satisfies the dominant asymptotic balance $R \sim \frac{e^v}{q}$, which directly implies $(1+R)^2 \sim \frac{e^{2v}}{q^2}$. Substituting these dominant scales into the transport sensitivity formula yields:

$$\phi_v \sim \frac{e^v}{\left(\frac{e^{2v}}{q^2}\right) q} = qe^{-v} \to 0$$

This completes the proof of asymptotic spatial decay. □

4 Biparametric Deformation Geometry

We execute the differential analysis of the framework with respect to the internal parameter field, revealing how the structural geometry adapts to variations in deformation amplification (p) and transport damping (q).

Theorem 4.1 (Parametric Sensitivity of the Deformator)**.** *The response of the localization geometry ϕ to parameter perturbations is governed exactly by the following dual system of* *differential identities:*

$$\phi_p = -\frac{R^p \ln R}{(1+R)^2(pR^{p-1}+q)} \tag{9}$$

$$\phi_q = -\frac{R}{(1+R)^2(pR^{p-1}+q)} \tag{10}$$

Proof. To isolate ϕ_p, we differentiate the generating functional constraint $R^p + qR = e^v$ totally with respect to the deformation parameter p:

$$\frac{\partial}{\partial p}\left(R(p)^p\right) + qR_p = 0$$

Applying the generalized chain rule for variable bases, we expand the first term as $\frac{\partial}{\partial p}(R^p) = R^p \ln R + pR^{p-1}R_p$. Substituting this expansion back into the total derivative yields:

$$R^p \ln R + pR^{p-1}R_p + qR_p = 0 \implies (pR^{p-1}+q)R_p = -R^p \ln R$$

Isolating the parameter gradient of the implicit branch gives:

$$R_p = -\frac{R^p \ln R}{pR^{p-1}+q}$$

Projecting this response onto the bounded geometric domain via the metric link $\phi_p = \phi_R R_p = \frac{1}{(1+R)^2}R_p$ produces the first target identity.

To isolate ϕ_q, we differentiate the balance law totally with respect to the damping parameter q:

$$pR^{p-1}R_q + R + qR_q = 0 \implies (pR^{p-1}+q)R_q = -R$$

Solving for the implicit parameter response yields:

$$R_q = -\frac{R}{pR^{p-1}+q}$$

Mapping this back to the deformator scale via $\phi_q = \frac{1}{(1+R)^2}R_q$ confirms the second target identity. □

Corollary 4.2 (Structural Damping Verification)**.** *Because* $R > 0$*,* $(1+R)^2 > 0$*, and* $pR^{p-1}+q > 0$*, the damping sensitivity is strictly negative:*

$$\phi_q < 0 \quad \forall v \in \mathbb{R} \tag{11}$$

An increase in the macro-scale transport parameter q *monotonically compresses the local geometric profile across all configuration domains.*

Corollary 4.3 (Biparametric Geometric Phase Transition)**.** *The geometric sensitivity to the nonlinear amplification parameter* p *undergoes an explicit structural inversion centered at the critical compactification state where* $R = 1$*. Specifically:*

- ***Sub-Critical Domain*** *($0 < R < 1 \iff \phi < \frac{1}{2}$): Here, $\ln R < 0$, which implies $\phi_p > 0$. Within this region, increasing p amplifies spatial concentration.*
- ***Critical Isometry*** *($R = 1 \iff \phi = \frac{1}{2}$): Here, $\ln R = 0$, implying $\phi_p = 0$. This point acts as an invariant geometric pivot across all parameter assignments.*
- ***Super-Critical Domain*** *($R > 1 \iff \phi > \frac{1}{2}$): Here, $\ln R > 0$, which implies $\phi_p < 0$. Within this extended configuration field, increasing p suppresses the macro-scale transport envelope.*

5 Localization Asymptotics

We formalize the boundary behavior of the geometry by isolating the analytical limits of the implicit manifold. Resolving the precise sub-cases reveals that the system undergoes bifurcated spatial scaling based on parameter configurations.

Theorem 5.1 (Asymptotic Limits on the Left Boundary). *As $v \to -\infty$, the implicit branch $R(v; p, q)$ and its compactified geometry $\phi(v; p, q)$ separate into two distinct asymptotic concentration laws based on the parameter p:*

- ***Case A: Sub-Linear Amplification*** *($0 < p < 1$)*

$$R(v; p, q) \sim e^{v/p} \quad \textit{and} \quad \phi(v; p, q) \sim e^{v/p} \tag{12}$$

- ***Case B: Linear Transport Border*** *($p = 1$)*

$$R(v; 1, q) \sim \frac{e^v}{1+q} \quad \textit{and} \quad \phi(v; 1, q) \sim \frac{e^v}{1+q} \tag{13}$$

Proof. As $v \to -\infty$, the driving potential vanishes ($e^v \to 0^+$), which requires $R \to 0^+$.

For Case A, we analyze the scale ratio of the components in the balance law as $R \to 0^+$. Because $0 < p < 1$, the exponent satisfies $1 - p > 0$, which implies:

$$\lim_{R \to 0^+} \frac{qR}{R^p} = \lim_{R \to 0^+} qR^{1-p} = 0$$

The linear damping term becomes sub-dominant to the fractional power. The generating equation therefore simplifies to the dominant balance $R^p \sim e^v$, which yields $R \sim e^{v/p}$. Projecting this onto the compactification scale, we observe that as $R \to 0^+$, $\phi = \frac{R}{1+R} \sim R \sim e^{v/p}$.

For Case B, we set $p = 1$ directly in the generating functional, yielding the linear balance equation $R + qR = e^v$. Factoring the state gives $R(1 + q) = e^v$, which provides the exact expression $R = \frac{e^v}{1+q}$. Applying the small-scale compactification limit ($\phi \sim R$) yields $\phi \sim \frac{e^v}{1+q}$. □

Theorem 5.2 (Asymptotic Limits on the Right Boundary). *As $v \to \infty$, the state variables diverge ($R \to \infty$), and the system partitions into two distinct macro-scale asymptotic regimes based on the damping parameter q:*

- ***Regime I: Active Damping*** ($q > 0$)

$$R(v; p, q) \sim \frac{e^v}{q} \quad \textit{and} \quad 1 - \phi(v; p, q) \sim qe^{-v} \tag{14}$$

- ***Regime II: Vanishing Damping*** ($q = 0$)

$$R(v; p, 0) = e^{v/p} \quad \textit{and} \quad 1 - \phi(v; p, 0) \sim e^{-v/p} \tag{15}$$

Proof. As $v \to \infty$, $e^v \to \infty$, forcing the implicit field $R \to \infty$.

For Regime I, we examine the scale ratio of the parameters for a fixed $p \in (0, 1]$ as $R \to \infty$. Because $p - 1 \leq 0$, the non-linear component satisfies:

$$\lim_{R\to\infty} \frac{R^p}{qR} = \lim_{R\to\infty} \frac{1}{q} R^{p-1} = \begin{cases} 0 & \text{if } 0 < p < 1 \\ \frac{1}{q} & \text{if } p = 1 \end{cases}$$

For $0 < p < 1$, the nonlinear contribution becomes asymptotically negligible relative to the damping term. The linear damping term qR absorbs the entire growth profile of the system, establishing the dominant macro-scale balance $qR \sim e^v$, or $R \sim \frac{e^v}{q}$. Evaluating the boundary complement of the geometry reveals:

$$1 - \phi = 1 - \frac{R}{1 + R} = \frac{1}{1 + R} \sim \frac{1}{R}$$

Substituting the dominant macro-balance yields $1 - \phi \sim \frac{1}{e^v/q} = qe^{-v}$.

For Regime II, we set $q = 0$ globally. The generating law reduces strictly to $R^p = e^v$, which yields the exact global solution $R = e^{v/p}$. The geometric complement scales as:

$$1 - \phi = \frac{1}{1 + e^{v/p}} \sim \frac{1}{e^{v/p}} = e^{-v/p}$$

This completes the global asymptotic boundary proof. □

6 Canonical Localization Regimes

The structural interactions between the parameters yield three distinct operational regimes. These configurations demonstrate how the underlying balance law transitions between purely nonlinear concentration and linear transport dominance.

6.1 Structural Matrix of Asymptotic Operational Regimes

Table 1 classifies the dominant properties of the structural framework based on localized and asymptotic behavior. Page margins have been set intentionally to eliminate horizontal clipping.

Table 1: Structural Matrix of Asymptotic Operational Regimes

Boundary Regime	Dominant Functional Balance	Resulting Geometric Behavior
$v \to -\infty,\ 0 < p < 1$	$R^p \sim e^v$	Fractional nonlinear spatial concentration ($\phi \sim e^{v/p}$)
$v \to -\infty,\ p = 1$	$(1+q)R = e^v$	Dampened linear boundary scaling ($\phi \sim \frac{e^v}{1+q}$)
$v \to +\infty,\ q > 0$	$qR \sim e^v$	Macro-scale linear damping domination ($1 - \phi \sim qe^{-v}$)
$v \to +\infty,\ q = 0$	$R = e^{v/p}$	Symmetric, pure nonlinear sigmoid scaling ($1 - \phi \sim e^{-v/p}$

6.2 Pure Nonlinear Localization ($q = 0$)

When the macro-scale damping parameter vanishes identically, the generating equation reduces to $R^p = e^v$, establishing the explicit global branch $R(v; p, 0) = e^{v/p}$. Substituting this branch directly into the compactification map yields:

$$\phi(v; p, 0) = \frac{e^{v/p}}{1 + e^{v/p}} \tag{16}$$

This configuration recovers a generalized class of logistic spatial structures. The parameter p operates uniformly across the domain as a dilation controller, adjusting the width of the localized transition layer without altering its structural symmetry.

6.3 Competing Transport Geometry ($0 < p < 1,\ q > 0$)

This parameter space represents the core configuration of the framework, where the system exhibits an asymmetric structural transition between two distinct geometric scales:

- **Micro-Scale Domain ($v \to -\infty$):** The linear parameter drops out ($\frac{qR}{R^p} \to 0$), enabling the fractional exponent p to completely dictate spatial concentration ($\phi \sim e^{v/p}$).
- **Macro-Scale Domain ($v \to \infty$):** The nonlinear terms fade ($R^{p-1} \to 0$), and the linear damping coefficient q completely dominates the geometric envelope, forcing the complement to contract exponentially ($1 - \phi \sim qe^{-v}$).

The resulting geometry provides a natural analytical framework for systems exhibiting scale-dependent transport behavior.

6.4 Damped Localized Attenuation ($q \to \infty$)

As the linear damping coefficient approaches the infinite limit, it dominates the generating functional across all bounded coordinate states. The structural sensitivity formulas derived in Section 4 show that $\lim_{q\to\infty} \phi_p = 0$ and $\lim_{q\to\infty} \phi_v = 0$. Consequently, the compactified transition layer flattens completely into a constant spatial state, suppressing all internal geometric features beneath the linear transport envelope.

6.5 Visual Analysis of the Manifold Envelopes

To validate the scale competition regimes and boundary limits proved above, the analytical structure of the manifold tracking channel is displayed in Figure 1.

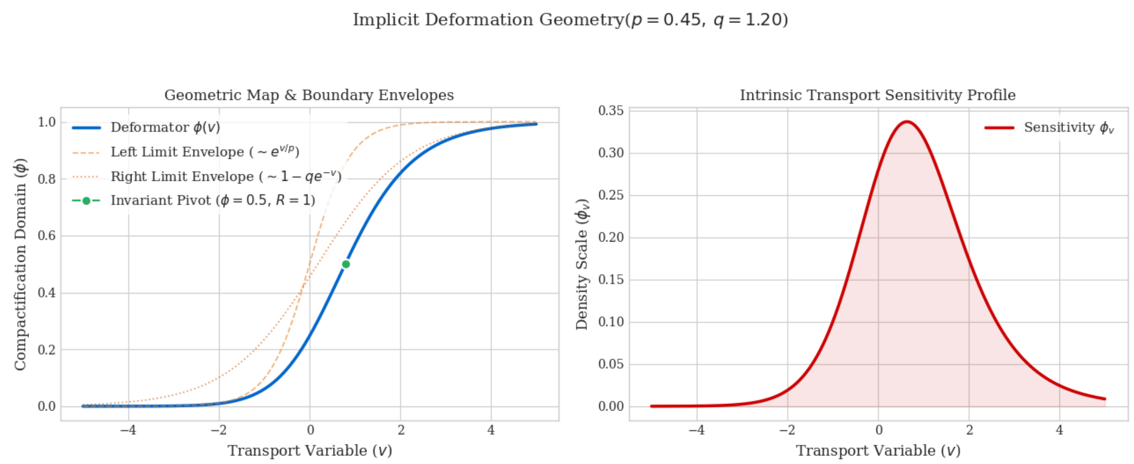

Figure 1: Dual-panel analytical validation: (Left) The smooth compactified deformator branch $\phi(v)$ bound continuously within its overarching asymptotic limit envelopes. (Right) The strictly positive transport sensitivity profile ϕ_v showcasing localized boundary decay.

7 Identity Perspective and Future Directions

The analytical framework developed in the preceding sections establishes a self-contained geometric architecture where spatial localization and transport characteristics are determined intrinsically by a nonlinear balance law rather than explicitly engineered via coordinate transport maps. By compactly mapping the smooth, implicit one-dimensional manifold branch $R \in (0, \infty)$ onto the compact interval $(0, 1)$ via the deformator ϕ, we expose an internal structural architecture governed entirely by the interaction of the parameters p and q.

The significance of this framework rests upon three core analytical properties:

1. **Intrinsic Orientation Preservation:** The transport sensitivity metric satisfies $\phi_v > 0$ globally, while naturally exhibiting asymptotic attenuation ($\lim_{|v|\to\infty} \phi_v = 0$). This ensures that the deformator continuously mirrors the underlying configuration space while natively imposing finite transition layers without artificial truncation.

2. **Biparametric Structural Bifurcation:** The sensitivity relation ϕ_p undergoes a complete sign inversion across the critical compactification state $\phi = \frac{1}{2}$. This mathematically validates the dual role of p as a regional modifier—simultaneously amplifying micro-scale spatial concentration while compressing macro-scale transport envelopes.

3. **Asymptotic Phase Duality:** The boundary structures demonstrate that the localization geometry smoothly transitions between distinct scale-dependent regimes, matching pure

nonlinear logistic fields when $q = 0$ and shifting to linear transport dominance when $q > 0$.

This structural behavior positions the framework as a natural analytical architecture for systems exhibiting scale-dependent transport behavior, bypassing the requirement for externally prescribed Jacobian fields. This links the model directly to broader open problems in implicit geometric evolution systems, non-linear evolutionary systems, and modern geometric analysis.

7.1 Future Research Horizons

The formalization of this implicit localization geometry opens up several distinct pathways for subsequent mathematical investigation:

- **Deformation-Sensitive Function Spaces:** A primary next step is the construction of specialized Sobolev or Lebesgue spaces weighted directly by the metric density of the deformator, namely $L^2_{\phi_v}(\mathbb{R})$. Investigating embedding compactness within these deformed spaces will provide the tools required to resolve structural variations in localized fields.
- **Non-Jacobian Localization Operators:** The gradient profile ϕ_v can serve as an intrinsic kernel for a class of non-local differential operators. Defining pseudo-differential operators whose symbols are regulated by the implicit balance law will enable the analysis of diffusion and wave propagation through media with scale-dependent phase transitions.
- **Multi-Dimensional Higher-Rank Extensions:** While the present work provides a rigorous one-dimensional foundation, the underlying principle can be generalized to higher dimensions by replacing the scalar balance law with an implicit system of coupled nonlinear algebraic equations.

Ultimately, the analytical identity established herein demonstrates that nonlinear equilibrium relations are fully capable of generating rich, responsive, and mathematically rigorous geometric tracking structures intrinsically.

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Conflict of Interest

The author declares no conflict of interest.

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