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Differential Semigroups of Automaton Perturbations and Incremental Syntactic Reconstruction

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Abstract

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ and $\mathcal{B} = (Q, \Sigma, \delta', q_0, F)$ be deterministic finite automata on the same state set and with the same input alphabet. For each word $w \in \Sigma^*$, let $\delta_w, \delta'_w \in Q^Q$ be the induced transformations. We introduce the differential transition semigroup

$$\mathscr{D}(\mathcal{A}, \mathcal{B}) = \{(\delta_w, \delta'_w) : w \in \Sigma^*\} \leq Q^Q \times Q^Q,$$

and the defect set $\mathrm{Def}(f, g) = \{q \in Q : f(q) \neq g(q)\}$. A composition law for defects is established and used to prove a backward localization theorem: the disagreement of every word-induced pair is contained in the backward saturation of the set of locally modified states. Consequently, if the initial state lies outside the backward saturation, then the recognized language is invariant and the syntactic semigroups coincide. A local update algorithm is then obtained whose running time depends on the saturation size rather than on the full transition semigroup. The construction yields a rigorous differential framework for dynamic automata and incremental syntactic reconstruction.

Keywords: syntactic semigroup; transformation semigroup; automaton perturbation; defect propagation; incremental computation; regular language.

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1 Introduction

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. The transition semigroup

$$T(\mathcal{A}) = \{\delta_w : w \in \Sigma^*\}$$

is the classical algebraic carrier of the dynamics of $\mathcal{A}$; the syntactic semigroup of the recognized language is its faithful quotient [4, 10, 6, 3, 11]. The algebraic theory of finite semigroups and regular languages is well developed [4, 10, 1, 2, 5]. What is not standard is a differential treatment of *automaton perturbation*: one starts with a base automaton $\mathcal{A}$, changes finitely many transition values, obtains a perturbed automaton $\mathcal{B}$, and asks for algebraic invariants and reconstruction procedures controlled by the perturbation rather than by the entire semigroup.

The present paper introduces a new differential object $\mathscr{D}(\mathcal{A}, \mathcal{B})$, a subsemigroup of the direct product $T(\mathcal{A}) \times T(\mathcal{B})$, together with a defect calculus. The main theorem proves that defects propagate only through a backward saturation of the set of locally modified states. This yields a structural stability theorem for languages and syntactic semigroups, and an incremental update algorithm. The semigroup-theoretic viewpoint is motivated by standard transformation semigroup methods [6, 3, 11] and is consistent with recent algebraic work on semigroups and computation [7, 9, 8].

2 Preliminaries

Fix a finite set Q and write Q^Q for the full transformation monoid on Q, with multiplication defined by composition $(f, g) \mapsto f \circ g$. For brevity, fg denotes $f \circ g$, so that $(fg)(q) = f(g(q))$.

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. For each $a \in \Sigma$, define $\delta_a(q) = \delta(q, a)$. For $w = a_1 \cdots a_n \in \Sigma^*$, let

$$\delta_w = \delta_{a_n} \cdots \delta_{a_1} \in Q^Q,$$

with $\delta_\varepsilon = \mathrm{id}_Q$. The transition semigroup $T(\mathcal{A})$ is the subsemigroup of Q^Q generated by $\{\delta_a : a \in \Sigma\}$.

The language recognized by $\mathcal{A}$ is

$$L(\mathcal{A}) = \{w \in \Sigma^* : \delta_w(q_0) \in F\}.$$

For $L \subseteq \Sigma^*$, the syntactic congruence is

$$u \equiv_L v \iff (\forall x, y \in \Sigma^*)\ xuy \in L \iff xvy \in L,$$

and the syntactic semigroup is $\mathrm{Syn}(L) = \Sigma^*/{\equiv_L}$ [4, 10, 1].

Throughout, two automata

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F), \qquad \mathcal{B} = (Q, \Sigma, \delta', q_0, F)$$

are assumed to have the same state set, alphabet, initial state and final-state set. Only the transition maps differ.

Definition 2.1. *For $w \in \Sigma^*$ let*

$$\partial_w = (\delta_w, \delta'_w) \in Q^Q \times Q^Q.$$

The differential transition semigroup *of the pair $(\mathcal{A}, \mathcal{B})$ is*

$$\mathscr{D}(\mathcal{A}, \mathcal{B}) = \{\partial_w : w \in \Sigma^*\} \subseteq Q^Q \times Q^Q.$$

For $f, g \in Q^Q$, define the defect set

$$\mathrm{Def}(f, g) = \{q \in Q : f(q) \neq g(q)\}.$$

Lemma 2.1. *For all $u, v \in \Sigma^*$,*

$$\partial_{uv} = \partial_u \partial_v.$$

Consequently,

$$\partial_a = (\delta_a, \delta'_a) \quad (a \in \Sigma)$$

generate $\mathscr{D}(\mathcal{A}, \mathcal{B})$.

Proof. Write $u = a_1 \cdots a_m$ and $v = b_1 \cdots b_n$. Then

$$\delta_{uv} = \delta_{b_n} \cdots \delta_{b_1} \delta_{a_m} \cdots \delta_{a_1} = \delta_v \delta_u,$$

and similarly $\delta'_{uv} = \delta'_v \delta'_u$. Since our multiplication is composition, this gives

$$\partial_{uv} = (\delta_{uv}, \delta'_{uv}) = (\delta_v \delta_u, \delta'_v \delta'_u) = \partial_u \partial_v.$$

Thus every element of $\mathscr{D}(\mathcal{A}, \mathcal{B})$ is a product of generators ∂_a with $a \in \Sigma$. □

Theorem 2.1. *The set $\mathscr{D}(\mathcal{A}, \mathcal{B})$ is a subsemigroup of $Q^Q \times Q^Q$. The coordinate projections*

$$\pi_1 : \mathscr{D}(\mathcal{A}, \mathcal{B}) \to T(\mathcal{A}), \qquad \pi_2 : \mathscr{D}(\mathcal{A}, \mathcal{B}) \to T(\mathcal{B}),$$

given by $\pi_1(f, g) = f$ and $\pi_2(f, g) = g$, are surjective homomorphisms. Hence $\mathscr{D}(\mathcal{A}, \mathcal{B})$ is a subdirect product of $T(\mathcal{A})$ and $T(\mathcal{B})$.

Proof. Closure under multiplication follows from Lemma 2.1. Since composition in $Q^Q \times Q^Q$ is associative, $\mathscr{D}(\mathcal{A}, \mathcal{B})$ is a semigroup. For surjectivity of π_1, let $f \in T(\mathcal{A})$. Then $f = \delta_w$ for some $w \in \Sigma^*$, and $\pi_1(\partial_w) = \delta_w = f$. Similarly π_2 is surjective. Coordinate projections preserve multiplication, so they are homomorphisms. □

3 Defect calculus

The central invariant is the defect set. We require a composition theorem.

Lemma 3.1. *For all* $f, g, u, v \in Q^Q$,

$$\mathrm{Def}(fu, gv) \subseteq \mathrm{Def}(u, v) \cup u^{-1}(\mathrm{Def}(f, g)).$$

Here

$$u^{-1}(X) = \{q \in Q : u(q) \in X\}.$$

Proof. Let $q \in Q \setminus \big(\mathrm{Def}(u, v) \cup u^{-1}(\mathrm{Def}(f, g))\big)$. Then $q \notin \mathrm{Def}(u, v)$ and $q \notin u^{-1}(\mathrm{Def}(f, g))$. The first condition gives $u(q) = v(q)$. Put $s = u(q) = v(q)$. The second condition implies $s \notin \mathrm{Def}(f, g)$, hence $f(s) = g(s)$. Therefore

$$(fu)(q) = f(u(q)) = f(s) = g(s) = g(v(q)) = (gv)(q).$$

Thus $q \notin \mathrm{Def}(fu, gv)$. Since every point outside the right-hand side lies outside $\mathrm{Def}(fu, gv)$, the inclusion follows. □

Corollary 3.1. *For* $u, v \in \Sigma^*$,

$$\mathrm{Def}(\delta_{uv}, \delta'_{uv}) \subseteq \mathrm{Def}(\delta_u, \delta'_u) \cup \delta_u^{-1}(\mathrm{Def}(\delta_v, \delta'_v)).$$

Proof. Apply Lemma 3.1 with $f = \delta_v$, $g = \delta'_v$, $u = \delta_u$, and $v = \delta'_u$. Since

$$\mathrm{Def}(\delta_u, \delta'_u) = \mathrm{Def}(u, v)$$

in the notation of the lemma, the result follows. □

To localize disagreement, define the set of locally modified states

$$M = \bigcup_{a \in \Sigma} \mathrm{Def}(\delta_a, \delta'_a).$$

Define a sequence of subsets of Q by

$$B_0 = M, \qquad B_{n+1} = B_n \cup \bigcup_{a \in \Sigma} \delta_a^{-1}(B_n) \qquad (n \geq 0).$$

Since Q is finite, there exists $r \leq |Q|$ such that $B_r = B_{r+1}$. Put

$$B_\infty = \bigcup_{n \geq 0} B_n = B_r.$$

Lemma 3.2. *For every* $n \geq 0$ *and every word* $w \in \Sigma^*$ *of length* n,

$$\mathrm{Def}(\delta_w, \delta'_w) \subseteq B_n.$$

In particular,

$$\mathrm{Def}(\delta_w, \delta'_w) \subseteq B_\infty \qquad (w \in \Sigma^*).$$

Proof. We argue by induction on $n = |w|$.

If $n = 0$, then $w = \varepsilon$ and $\delta_w = \delta'_w = \mathrm{id}_Q$, so

$$\mathrm{Def}(\delta_w, \delta'_w) = \varnothing \subseteq B_0.$$

Assume the statement holds for all words of length n. Let $w = ua$ with $|u| = n$ and $a \in \Sigma$. By Corollary 3.1,

$$\mathrm{Def}(\delta_{ua}, \delta'_{ua}) \subseteq \mathrm{Def}(\delta_u, \delta'_u) \cup \delta_u^{-1}(\mathrm{Def}(\delta_a, \delta'_a)).$$

By the induction hypothesis, $\mathrm{Def}(\delta_u, \delta'_u) \subseteq B_n$. Also $\mathrm{Def}(\delta_a, \delta'_a) \subseteq M = B_0 \subseteq B_n$, since the sequence B_n is increasing. Hence

$$\delta_u^{-1}(\mathrm{Def}(\delta_a, \delta'_a)) \subseteq \delta_u^{-1}(B_n).$$

Write $u = a_1 \cdots a_n$. Then

$$\delta_u^{-1}(B_n) = \delta_{a_1}^{-1} \cdots \delta_{a_n}^{-1}(B_n) \subseteq B_{n+1},$$

because each preimage under a generator is adjoined in the recursive definition of B_{n+1}. Therefore

$$\mathrm{Def}(\delta_{ua}, \delta'_{ua}) \subseteq B_n \cup B_{n+1} = B_{n+1}.$$

The inductive step is proved. □

Remark 3.1. *Lemma 3.2 is the differential localization principle: all disagreement is forced to remain inside the backward orbit of the local modification set M.*

The next theorem is the new deeper structural result. It identifies the maximal region on which every perturbation word acts identically.

Theorem 3.1 (Maximal unaffected region)**.** *Let*

$$U = Q \setminus B_\infty.$$

Then the following hold:

(i) $U \cap M = \varnothing$.

(ii) U *is forward invariant under every generator of* $\mathcal{A}$ *and* $\mathcal{B}$*; that is,*

$$\delta_a(U) \subseteq U, \qquad \delta'_a(U) \subseteq U \qquad (a \in \Sigma).$$

(iii) For every word $w \in \Sigma^*$,

$$\delta_w|_U = \delta'_w|_U.$$

(iv) If $R \subseteq Q$ *satisfies* $\delta_w|_R = \delta'_w|_R$ *for all* $w \in \Sigma^*$*, then* $R \subseteq U$.

Hence U is the unique maximal subset of Q on which all word-induced transformations of $\mathcal{A}$ and $\mathcal{B}$ agree.

Proof. (i) Since $B_0 = M \subseteq B_\infty$, we have $U \cap M = \varnothing$.

(ii) Let $q \in U$ and $a \in \Sigma$. Suppose $\delta_a(q) \in B_\infty$. Then, by definition of preimage,

$$q \in \delta_a^{-1}(B_\infty).$$

Because B_∞ is stable under taking δ_a-preimages, one has $\delta_a^{-1}(B_\infty) \subseteq B_\infty$. Hence $q \in B_\infty$, contradicting $q \in U$. Therefore $\delta_a(q) \in U$, so $\delta_a(U) \subseteq U$.

Now let $q \in U$. By (i), $q \notin M$, and therefore $q \notin \mathrm{Def}(\delta_a, \delta'_a)$ for every $a \in \Sigma$. Thus

$$\delta_a(q) = \delta'_a(q).$$

Since $\delta_a(q) \in U$, it follows that $\delta'_a(q) = \delta_a(q) \in U$. Hence $\delta'_a(U) \subseteq U$.

(iii) We prove by induction on $|w|$. If $w = \varepsilon$, then both restrictions equal id_U. Assume the claim holds for words of length n, and let $w = ua$ with $|u| = n$ and $a \in \Sigma$. Take $q \in U$ and put $s = \delta_u(q)$. By the induction hypothesis, $\delta_u(q) = \delta'_u(q)$; by (ii), $s \in U$. Again by (ii),

$$\delta_a(s) = \delta'_a(s).$$

Therefore

$$\delta_w(q) = \delta_a(\delta_u(q)) = \delta_a(s) = \delta'_a(s) = \delta'_a(\delta'_u(q)) = \delta'_w(q).$$

So $\delta_w|_U = \delta'_w|_U$.

(iv) Let $R \subseteq Q$ satisfy $\delta_w|_R = \delta'_w|_R$ for all $w \in \Sigma^*$. Suppose that $r \in R \cap B_\infty$. Since B_∞ is the least subset of Q containing M and closed under all preimages δ_a^{-1}, there exist $n \geq 0$, symbols $a_1, \ldots, a_n \in \Sigma$, and a state $m \in M$ such that

$$\delta_{a_n} \cdots \delta_{a_1}(r) = m.$$

Let $u = a_1 \cdots a_n$. Then $\delta_u(r) = m \in M$. Choose $b \in \Sigma$ with $m \in \mathrm{Def}(\delta_b, \delta'_b)$. Put $w = ub$. Since $\delta_u(r) = \delta'_u(r) = m$, we get

$$\delta_w(r) = \delta_b(m) \neq \delta'_b(m) = \delta'_w(r),$$

contradicting $\delta_w|_R = \delta'_w|_R$. Thus $R \cap B_\infty = \varnothing$, and hence $R \subseteq U$.

Therefore U is maximal with the stated property. □

Corollary 3.2. *If $q_0 \notin B_\infty$, then*

$$L(\mathcal{A}) = L(\mathcal{B}).$$

Consequently,

$$\mathrm{Syn}(L(\mathcal{A})) \cong \mathrm{Syn}(L(\mathcal{B})).$$

Proof. By Lemma 3.2, $q_0 \notin \mathrm{Def}(\delta_w, \delta'_w)$ for every $w \in \Sigma^*$. Hence

$$\delta_w(q_0) = \delta'_w(q_0) \qquad (w \in \Sigma^*).$$

Since both automata have the same final-state set F,

$$\delta_w(q_0) \in F \iff \delta'_w(q_0) \in F,$$

so $L(\mathcal{A}) = L(\mathcal{B})$. Equal languages have equal syntactic congruences; therefore their syntactic semigroups are canonically isomorphic [4, 10, 1]. □

4 Incremental reconstruction

Let $R = Q \setminus B_\infty$. By Corollary 3.2, states in R are irrelevant to language change whenever the initial state lies in R. More generally, only the restricted dynamics on B_∞ need be recomputed.

For each $a \in \Sigma$, define the localized map

$$\overline{\delta}'_a = \delta'_a|_{B_\infty} : B_\infty \to Q.$$

The update algorithm computes the semigroup generated by $\{\overline{\delta}'_a : a \in \Sigma\}$ on B_∞, while keeping the old transformations on R unchanged.

Proposition 4.1. *Assume the tables of δ_a and δ'_a are given for all $a \in \Sigma$. Then:*

(i) B_∞ can be computed in time $O(|\Sigma|\,|Q|^2)$.

(ii) The restricted transition semigroup of $\mathcal{B}$ on B_∞ can be generated by a breadth-first closure procedure in time

$$O(|\Sigma|\,|B_\infty|\,N),$$

where N is the number of distinct restricted transformations obtained.

(iii) If $|B_\infty| \ll |Q|$, then the update cost is controlled by the perturbation zone rather than by the size of $T(\mathcal{B})$ on all of Q.

Proof. (i) To compute M, one compares $\delta_a(q)$ and $\delta'_a(q)$ for each $(q, a) \in Q \times \Sigma$, which costs $O(|\Sigma|\,|Q|)$. Then each iteration

$$B_{n+1} = B_n \cup \bigcup_{a \in \Sigma} \delta_a^{-1}(B_n)$$

can be computed by scanning all pairs (q, a) and checking whether $\delta_a(q) \in B_n$. This costs $O(|\Sigma|\,|Q|)$ per iteration. Since the increasing chain stabilizes after at most $|Q|$ iterations, the total cost is $O(|\Sigma|\,|Q|^2)$.

(ii) Represent a restricted transformation $f : B_\infty \to B_\infty \cup R$ by its value table on B_∞. Start with the set of generators $\{\overline{\delta}'_a : a \in \Sigma\}$ and repeatedly compose by generators until no new table appears. Each composition of two tables costs $O(|B_\infty|)$. If N distinct tables are produced and each is multiplied by $|\Sigma|$ generators, the total cost is $O(|\Sigma|\,|B_\infty|\,N)$.

(iii) This follows immediately from (ii), because the factor $|B_\infty|$ replaces the global state size $|Q|$ in each composition. □

5 Syntactic transfer

The differential semigroup also controls the syntactic quotient.

Theorem 5.1. *Let*

$$\eta_{\mathcal{A}} : \Sigma^* \to T(\mathcal{A}), \qquad \eta_{\mathcal{B}} : \Sigma^* \to T(\mathcal{B})$$

be the canonical morphisms $\eta_{\mathcal{A}}(w) = \delta_w$*,* $\eta_{\mathcal{B}}(w) = \delta'_w$*. Define*

$$\eta : \Sigma^* \to \mathscr{D}(\mathcal{A}, \mathcal{B}), \qquad \eta(w) = \partial_w.$$

Then η *is a surjective semigroup morphism, and*

$$\ker \eta = \ker \eta_{\mathcal{A}} \cap \ker \eta_{\mathcal{B}}.$$

In particular, $\mathscr{D}(\mathcal{A}, \mathcal{B})$ *is a quotient of the free semigroup* Σ^+ *by the intersection of the two transition congruences.*

Proof. For $u, v \in \Sigma^*$, Lemma 2.1 yields

$$\eta(uv) = \partial_{uv} = \partial_u \partial_v = \eta(u)\eta(v),$$

so η is a semigroup morphism. Surjectivity is immediate from Definition 2.1.

Now let $u, v \in \Sigma^*$. Then

$$\eta(u) = \eta(v) \iff (\delta_u, \delta'_u) = (\delta_v, \delta'_v) \iff \delta_u = \delta_v \text{ and } \delta'_u = \delta'_v.$$

The latter is equivalent to $(u, v) \in \ker \eta_{\mathcal{A}}$ and $(u, v) \in \ker \eta_{\mathcal{B}}$, that is,

$$(u, v) \in \ker \eta_{\mathcal{A}} \cap \ker \eta_{\mathcal{B}}.$$

Hence $\ker \eta = \ker \eta_{\mathcal{A}} \cap \ker \eta_{\mathcal{B}}$. □

Corollary 5.1. *If* $L(\mathcal{A}) = L(\mathcal{B})$*, then the syntactic congruence of the common language contains both transition congruences and therefore contains* $\ker \eta$*.*

Proof. If $L(\mathcal{A}) = L(\mathcal{B}) = L$, the syntactic congruence $\equiv_L$ identifies words inducing the same transformation in the minimal automaton of L [4, 10, 5]. Hence it contains each transition congruence, and therefore also their intersection $\ker \eta$. □

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Conflict of Interest

The author(s) declare no conflict of interest.

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