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Exact regret envelopes for commitment-constrained manufacturing under joint capacity stress

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Abstract

A feasible manufacturing adjustment can preserve mandatory commitments while losing most of the contribution attainable through reoptimization. This paper studies nominal-surplus scaling under simultaneous capacity losses and increasing production floors. For arbitrary finite product and resource counts with nonnegative resource use, it derives the exact feasible stress interval and a finite breakpoint audit of the largest scaled-plan regret. A two-product family with strictly positive resource coefficients and positive commitments proves that the optimum-to-scaled-value ratio can grow without bound. A dual-certificate tracing procedure supplies an interval-wide regret bracket without enumerating every dual vertex. The generic tracing principle is established parametric optimization; its specialization here couples commitment-adjusted scaling with a complete regret audit. An explicit two-resource model also gives exact product-mix transitions, shadow prices and coefficient-uncertainty limits. Experiments cover 90 manufacturing instances and compare oracle tracing, direct optimizer grids and independently enumerated dual envelopes. The trace requires a mean of 3.58 LP queries on these instances; 62 have an interior regret maximum, and a 101-point grid understates the largest regret by up to 0.4425 accounting units. These synthetic results support complete-path auditing rather than extrapolation from a nominal shadow price.

Keywords: product mix; manufacturing; capacity stress; mandatory commitments; shadow price; linear programming; robust feasibility.

1. Introduction and research question

Robust multi-site production planning already has substantial precedents [14, 15]. The contribution developed here is a fully solved parametric manufacturing family, with exact feasibility, product-mix transitions and certificates; it is not the invention of robust production planning.

A solver may return a profitable production schedule without explaining how long that schedule remains appropriate as resources tighten. A manufacturer needs more than the optimum at one capacity level. Useful questions include the smallest feasible capacity, the point at which a product mix changes, the marginal profit associated with another unit of capacity, and the loss from applying a simple rule instead of reoptimization.

The paint-production study of Okeke and Akpan [20] gives a direct resource-allocation foundation. The medication-production study of Okeke and Nwokolo [22] provides another manufacturing application of limited inputs and competing outputs. The present products are generic industrial products A and B; they are not medicines assigned to patients. No coefficients or numerical outcomes from those publications are represented as measurements in this manuscript.

Distribution connects production commitments to customers. The harmonic-mean transportation study [19] motivates keeping dispatch decisions distinct from manufacturing decisions. The coronavirus-sensitivity work [21] is included as a mathematical application that motivates careful terminology for sensitivity. Its time-derivative and disease interpretations are not imported. Here sensitivity means a change in optimized manufacturing profit with respect to resource capacity or a commitment parameter. All equations below concern quantities, capacities and accounting contributions.

Linear-programming theory already supplies optimality and sensitivity tools [9, 5, 25]. The research question is whether a feasible adjustment of a nominal optimum can be audited over a complete capacity–commitment stress path. The paper derives a general regret partition, proves that nominal-surplus scaling can have unbounded relative loss, and specializes an established certificate-tracing method to audit that loss. A two-product family gives hand-checkable transition formulas; 90 higher-dimensional examples supply independent envelope comparisons.

2. General production model and feasibility envelope

Let $x \in \mathbb{R}^n$ be a production vector, $c \in \mathbb{R}^n$ its unit contribution vector, $A \in \mathbb{R}_+^{m \times n}$ its nonnegative resource-use matrix, $b \in \mathbb{R}_+^m$ available capacities, and $\ell \in \mathbb{R}_+^n$ mandatory minimum production. Consider

$$V(b, \ell) = \max\{c^T x : Ax \leq b, x \geq \ell\}. \quad (1)$$

The commitments are lower bounds, not an upper demand curve. The model assumes all additional feasible production can be sold at the stated contribution. When this assumption fails, demand ceilings or inventory costs must be added. Continuous variables describe divisible quantities; whole-batch restrictions require integer programming.

Proposition 2.1 (Exact feasibility under nonnegative resource use). *The feasible set of (1) is nonempty if and only if $A\ell \leq b$.*

Proof. If $x \geq \ell$, nonnegativity of A gives $Ax \geq A\ell$. Therefore feasibility implies $A\ell \leq b$. Conversely, when this inequality holds, $x = \ell$ is feasible. $\square$

This simple test is an exact theorem only for the stated model. Additional coupling constraints, forbidden routes, upper product bounds or negative matrix coefficients can invalidate it. This point will matter when production and distribution are integrated.

The substitution $z = x - \ell \geq 0$ gives

$$V = c^T \ell + \max\{c^T z : Az \leq b - A\ell, z \geq 0\}. \quad (2)$$

It exposes residual capacity after commitments. The resource vector $b - A\ell$ is directly auditable, whereas a solver infeasibility message alone may obscure which commitment exhausted which input.

Proposition 2.2 (Certificate with mandatory lower bounds). *Let x be feasible, $y \geq 0$, and $\mu = A^T y - c \geq 0$. Then*

$$c^T x \leq y^T b - \mu^T \ell. \quad (3)$$

Equality holds if $y^T (b - Ax) = 0$ and $\mu^T (x - \ell) = 0$, and then x is optimal.

Proof. Using $c = A^T y - \mu$ gives $c^T x = y^T Ax - \mu^T x \leq y^T b - \mu^T \ell$. Equality in both nonnegative slack terms establishes the claim by weak duality. $\square$

This is the ordinary LP dual after shifting the lower bounds, not a new duality theorem. Convex analysis [6] and polyhedral optimization [23] provide the broader foundations. Numerical methods [18] still require explicit tolerances when certificates are checked in floating point.

3. An exactly solvable two-product stress family

Set

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 8 \\ 10 \end{pmatrix}, \quad b = \begin{pmatrix} 120 \\ u \end{pmatrix}, \quad \ell(\rho) = \begin{pmatrix} 20 + 10\rho \\ 15 + 15\rho \end{pmatrix}, \quad (4)$$

where $0 \leq \rho \leq 1$ and $50 \leq u \leq 100$. Every number is synthetic. Resource 1 has fixed capacity 120; resource 2 has stressed capacity u . The parameter ρ increases two mandatory commitments. Quantities and accounting contributions use normalized units.

Proposition 2.1 yields

$$55 + 35\rho \leq 120, \quad 50 + 40\rho \leq u. \quad (5)$$

The first inequality holds throughout the stated ρ interval. Hence the complete feasible domain is

$$\mathcal{F} = \{(u, \rho) : 0 \leq \rho \leq 1, 50 + 40\rho \leq u \leq 100\}. \quad (6)$$

Define $L = 15 + 15\rho$ and $u_c(\rho) = 82.5 + 22.5\rho$.

Theorem 3.1 (Exact product-mix transition). *For every $(u, \rho) \in \mathcal{F}$, the unique optimal plan and its value are*

$$x^*(u, \rho) = \begin{cases} (u - 2L, L)^T, & u \leq u_c(\rho), \\ ((240 - u)/3, (2u - 120)/3)^T, & u \geq u_c(\rho), \end{cases} \quad (7)$$

$$V(u, \rho) = \min\{8u - 90 - 90\rho, 240 + 4u\}. \quad (8)$$

At the transition the two formulas give the same plan and value. If $u_c > 100$, only the first branch occurs in the permitted capacity interval.

Proof. For $u \leq u_c$, the plan $(u - 2L, L)$ has its second resource and second product floor active. The first-resource usage is $2u - 3L \leq 120$ precisely when $u \leq (120 + 3L)/2 = u_c$. Its first product is at least $20 + 10\rho$ precisely when $u \geq 50 + 40\rho$. Use $y = (0, 8)^T$ and $\mu = (0, 6)^T$ in Proposition 2.2. The bound is $8u - 6L$, attained by this plan. Since $\mu_2 > 0$, any optimizer must have $x_2 = L$; since $y_2 > 0$, it must have $x_1 + 2x_2 = u$. These requirements fix the plan uniquely.

For $u \geq u_c$, the intersection of both resource boundaries gives the second formula. Its second product is at least L . Its first product is at least $140/3 > 30$, since $u \leq 100$, and therefore exceeds its mandatory floor. Use $y = (2, 4)^T$ and $\mu = 0$. The attained dual bound is $240 + 4u$. Both positive dual components force both resource equations to hold at any optimizer, fixing the unique intersection. The branch values coincide at u_c . Their minimum selects the lower feasible upper bound on the respective side, giving (8). $\square$

The theorem supplies an explicit joint envelope rather than a single nominal optimum. At $\rho = 0$, the transition is $u_c = 82.5$. At $\rho = 1$, it is 105, outside the capacity range; a mandatory B output then remains active throughout the feasible domain.

4. Capacity sensitivities, commitment sensitivities and finite losses

Away from the transition,

$$\partial_u V = \begin{cases} 8, & u < u_c, \\ 4, & u > u_c, \end{cases} \quad \partial_\rho V = \begin{cases} -90, & u < u_c, \\ 0, & u > u_c. \end{cases} \quad (9)$$

At the transition the value is continuous and has distinct one-sided derivatives. A two-sided ordinary derivative is not available. Its concavity in u follows directly from (8); the capacity shadow prices lie in $[4, 8]$ at the kink. These are supergradients of a concave value function, not gradients of a biological trajectory.

For fixed ρ and feasible $u_1 < u_2$, integration of the two constant slopes gives

$$4(u_2 - u_1) \leq V(u_2, \rho) - V(u_1, \rho) \leq 8(u_2 - u_1). \quad (10)$$

The exact loss is obtained by splitting the interval at u_c if it crosses the transition. A nominal shadow price of four therefore underestimates capacity loss after the lower-floor regime is reached. A Taylor approximation using the nominal active set is valid only until that active set changes.

For feasible $\rho_1 < \rho_2$ at fixed u ,

$$0 \leq V(u, \rho_1) - V(u, \rho_2) \leq 90(\rho_2 - \rho_1).$$

This loss arises because additional mandatory production can force an economically less favourable mix. It does not imply that meeting a customer obligation lacks value. The objective simply does not include penalties for failing commitments, since failure is excluded by feasibility.

4.1. Regret of a scaled nominal surplus plan

At $\rho = 0$, the nominal plan for $u = 100$ is $(140/3, 80/3)$. Its surplus over the floor $(20, 15)$ is $(80/3, 35/3)$, using additional resources $(65, 50)$. A common feasible adjustment is to scale this surplus uniformly:

$$x^{\text{scale}}(u) = (20, 15)^T + \frac{u - 50}{50} (80/3, 35/3)^T, \quad 50 \leq u \leq 100. \quad (11)$$

Its first-resource usage is $55 + 65(u - 50)/50 \leq 120$ and its second-resource usage is exactly u . Its value is $310 + 6.6(u - 50) = 6.6u - 20$. Thus its exact regret relative to reoptimization is

$$\mathcal{R}(u) = \begin{cases} 1.4u - 70, & 50 \leq u \leq 82.5, \\ 260 - 2.6u, & 82.5 \leq u \leq 100. \end{cases} \quad (12)$$

The regret is zero at both endpoints and reaches 45.5 at the transition. This is a specific comparison with an explicitly defined feasible rule. It is not a universal bound for all adjustment heuristics.

5. Bounded coefficient uncertainty

Let each resource coefficient satisfy $A_{ij} \in [A_{ij}^0, A_{ij}^0 + \theta D_{ij}]$ and each capacity satisfy $b_i \in [b_i^0 - \theta d_i, b_i^0]$, where $D, d \geq 0$ and $\theta \geq 0$. A static nonnegative production plan meets every realization precisely when

$$(A^0 + \theta D)x \leq b^0 - \theta d. \quad (13)$$

Necessity follows by choosing the admissible simultaneous endpoints; sufficiency follows from monotonicity. This is the classical box-robust endpoint reduction [24, 3]. More selective uncertainty budgets can avoid assuming every coefficient simultaneously attains its endpoint [4], but they change the feasible set and require a different derivation.

With a fixed commitment vector ℓ , robust feasibility holds exactly when

$$\theta \leq \theta_{\max} := \min_{i: d_i + (D\ell)_i > 0} \frac{b_i^0 - (A^0\ell)_i}{d_i + (D\ell)_i}, \quad (14)$$

provided the nominal floor is feasible. Rows with zero denominator place no additional uncertainty restriction. The result follows by evaluating (13) at the componentwise smallest admissible plan $x = \ell$ and rearranging each row.

For $b^0 = (120, 100)$, $D = 0.1A^0$ and $d = (20, 20)$, the two thresholds are

$$\frac{65 - 35\rho}{25.5 + 3.5\rho}, \quad \frac{50 - 40\rho}{25 + 4\rho}.$$

At $\rho = 1$, the second resource limits θ to $10/29 \approx 0.344828$. An uncertainty radius of 0.5 would therefore invalidate the mandatory commitments, even though the nominal plan is feasible. No probability interpretation is attached to these endpoint intervals.

6. Computation and verification

The supplied script evaluates a grid of (u, ρ) values with SciPy linear programming and compares the result with (7)–(8). It saves the feasible grid and checks primal slacks, dual inequalities and the objective equality. Analytic formulas are the main verification: no optimization method can override an incorrect model specification.

7. Downstream allocation and relevant literature

Production commitments can be generated by a customer-distribution layer. Hitchcock's transportation model [12], network-flow formulations [10, 1], and stepping-stone improvement [8]

Table 1: Selected synthetic cases, independently checked by linear programming.

u	ρ	x_A	x_B	V	y_1	y_2
50	0	20	15	310	0	8
70	0	40	15	470	0	8
82.5	0	52.5	15	570	0	8
100	0	46.6667	26.6667	640	2	4
80	0.5	35	22.5	505	0	8
100	0.5	46.6667	26.6667	640	2	4
90	1	30	30	540	0	8
100	1	40	30	620	0	8

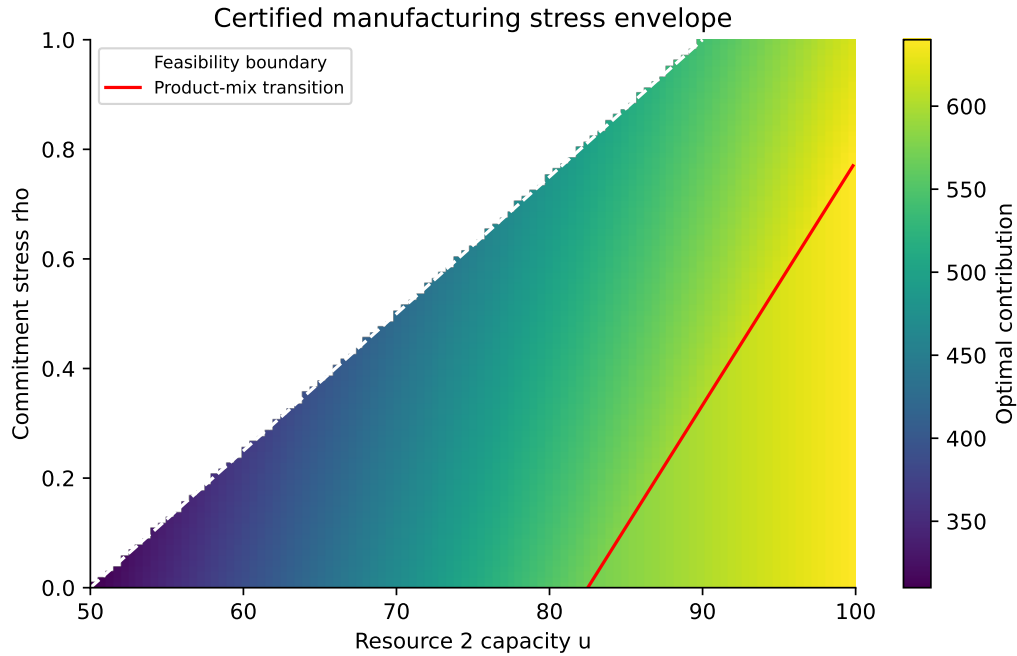


Figure 1: Exact optimized value over the feasible capacity–commitment domain. Blank cells are infeasible, rather than zero-profit plans.

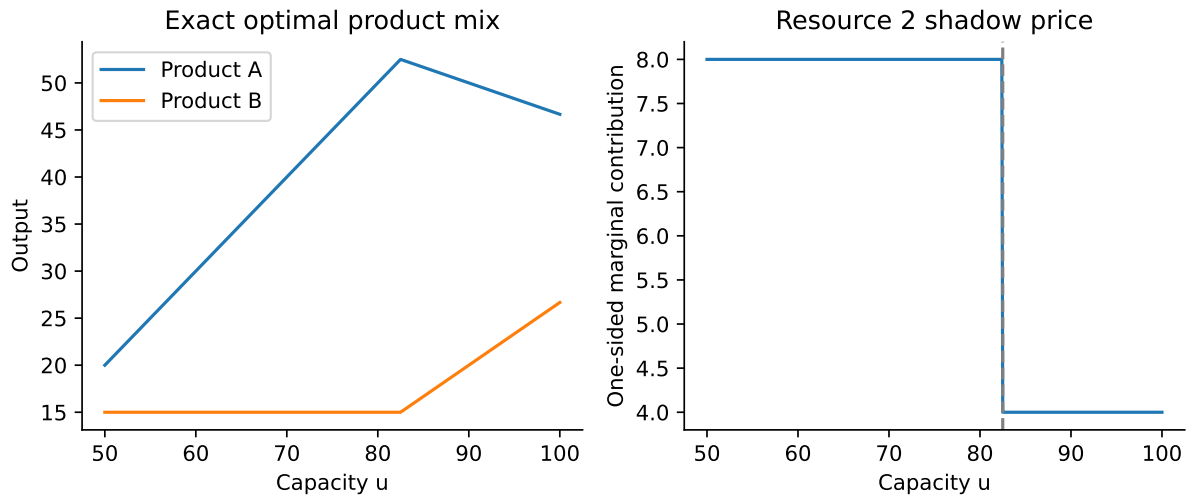


Figure 2: Product-mix and shadow-price changes under capacity stress at $\rho = 0$. The transition occurs at $u = 82.5$.

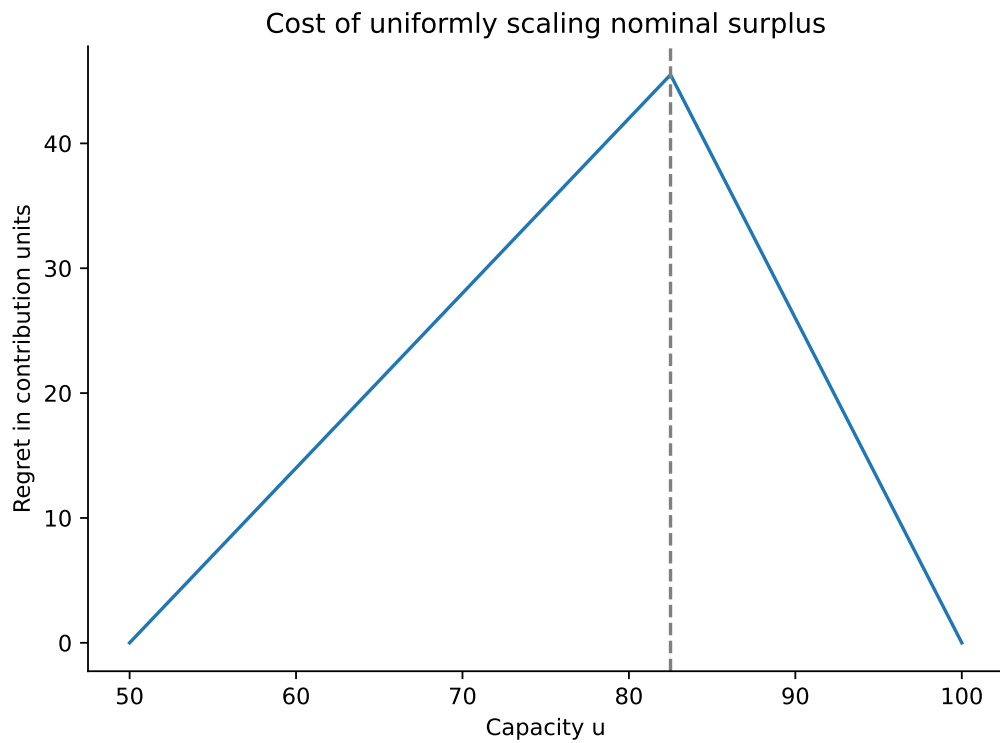


Figure 3: Loss from the feasible scaled-surplus rule relative to exact reoptimization. The maximum is 45.5 accounting units.

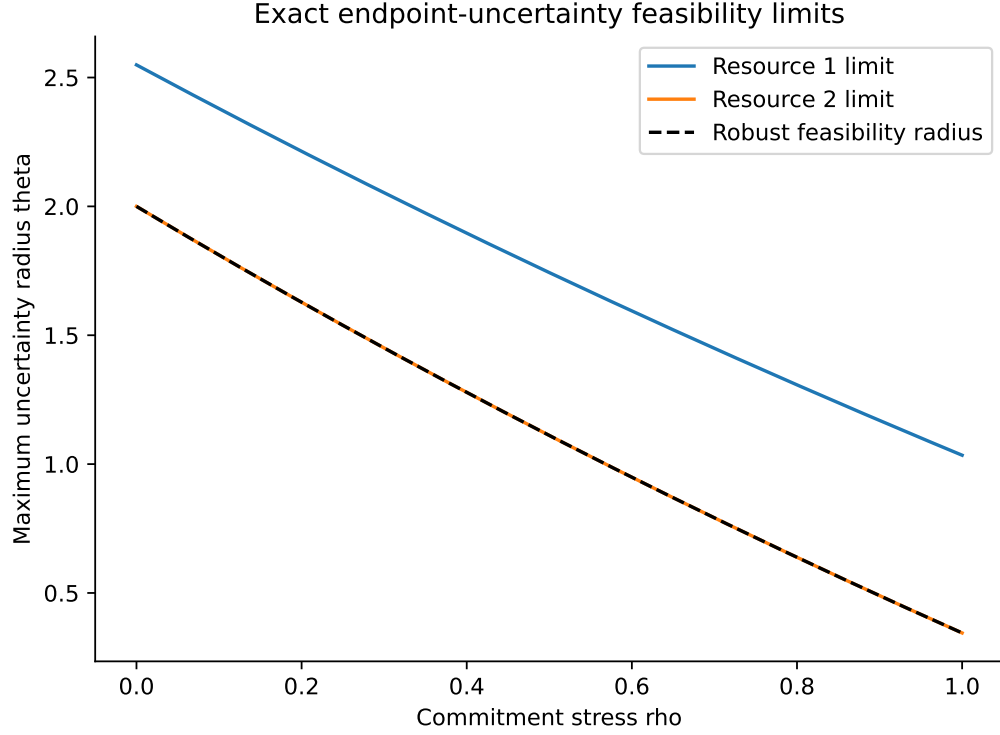


Figure 4: Exact robust-feasibility radius as commitments increase. The plot describes manufacturing capacities, not a biological threshold.

give appropriate downstream tools. The harmonic-mean application [19] supplies a possible initial dispatch schedule, but it should be checked or improved before its cost is included in a production profit estimate.

If finished lots must be assigned one-to-one to loading slots or contracts, the appropriate special case is assignment. Kuhn [13], Munkres [16], and Burkard et al. [7] provide exact assignment methods. These sources are relevant to that discrete delivery subproblem, not proofs of the manufacturing formulas. A full integrated model should optimize both stages jointly; independently maximizing gross production contribution need not maximize contribution after distribution costs.

The four selected application papers are used within their actual scope. Paint production [20] and medication manufacture [22] motivate constrained product mixes. Transportation [19] motivates delivery commitments. The sensitivity application [21] motivates declaring which variable is differentiated. None establishes the new numerical family, and none is cited as a biological mechanism in this industrial study. The rest of the bibliography concerns LP, assignment, networks, numerical optimization or robust optimization.

8. General joint-stress paths and regret envelopes

A complete research contribution should extend beyond the two-product illustration. Fix $A \in \mathbb{R}_+^{m \times n}$, $c \in \mathbb{R}^n$, and consider simultaneous capacity losses and growing commitments:

$$b(t) = b^0 - td, \quad \ell(t) = \ell^0 + te, \quad d, e \geq 0, \quad t \geq 0. \quad (15)$$

Assume $\ell^0 \geq 0$, $r^0 = b^0 - A\ell^0 \geq 0$, and finite attained optimal production values on the parameter interval of interest. Strictly positive resource coefficients and finite capacities are a sufficient

boundedness condition. Write $h = d + Ae$. The largest feasible stress along this ray is

$$T = \min_{i:h_i>0} \frac{r_i^0}{h_i}. \quad (16)$$

If all h_i vanish, there is no finite ray limit from these constraints; the study then specifies a finite analysis horizon separately. For $T < \infty$, the feasibility equivalence in Proposition 1 proves (16) exactly. The model may be applied after fixing a robust uncertainty radius and replacing A, b^0 with their valid worst endpoints.

Choose a feasible nominal surplus $z^0 = x^0 - \ell^0 \geq 0$. Define $s = Az^0$ and

$$\alpha(t) = \min \left\{ 1, \min_{i:s_i>0} \frac{r_i^0 - th_i}{s_i} \right\}, \quad x^S(t) = \ell^0 + te + \alpha(t)z^0, \quad 0 \leq t \leq T. \quad (17)$$

The minimum over an empty row set is omitted. Since all residual capacities are nonnegative, $0 \leq \alpha \leq 1$. For rows with $s_i = 0$, surplus scaling consumes nothing, so their residual feasibility is enough. Consequently x^S is feasible throughout the ray, even with nonuniform losses and growing commitments. When x^0 is a nominal optimizer, this rule reuses its composition but cannot generally preserve its optimality.

Let $Y = \{y \geq 0 : A^T y \geq c\}$. This pointed polyhedron has finitely many vertices, denoted $\mathcal{Y}$, and is nonempty under the boundedness assumptions. Standard LP duality yields

$$V(t) = c^T(\ell^0 + te) + \min_{y \in \mathcal{Y}} y^T(r^0 - th). \quad (18)$$

Thus V is a finite concave piecewise-affine function. The scaled value $S(t) = c^T x^S(t)$ is also piecewise affine, though concavity of S requires $c^T z^0 \geq 0$. The following regret result needs only piecewise affinity.

Theorem 8.1 (Finite certificate for worst scaled-plan regret). *Let K_V contain all breakpoints of (18), and let K_α contain all breakpoints of (17). Then*

$$\max_{0 \leq t \leq T} \{V(t) - S(t)\} = \max_{t \in \{0, T\} \cup K_V \cup K_\alpha} \{V(t) - S(t)\}. \quad (19)$$

The regret is nonnegative and continuous. Between consecutive points of the stated union its slope is

$$\frac{d}{dt}(V - S) = -y^T h - \alpha'(t)c^T z^0, \quad (20)$$

where y is an active optimal dual vertex. Hence a maximum can occur inside the feasible stress interval, and endpoint-only checks cannot certify the largest loss.

Proof. Feasibility of x^S gives $S \leq V$. Both values are continuous finite envelopes of affine functions. On each interval of the union partition, the same affine branch of V and the same affine branch of α apply. Their difference is affine and attains a maximum at an interval endpoint, including constant branches. Compactness gives the finite maximum over the union. Differentiating the two active branches cancels the common term $c^T e$ and gives (20). $\square$

The theorem is a specialization of established parametric LP geometry [11, 17]. Its decision content is the complete regret audit for the commitment-adjusted scaling policy. It does not concern stochastic online regret, nor assert a polynomial bound on the number of breakpoints. Murty's complexity result [17] prevents an unqualified scalability claim for unrestricted parametric LP.

8.1. An unbounded relative-loss family with positive commitments

The regret audit can detect more than a modest numerical loss. A feasible scaling policy can have no finite multiplicative performance guarantee even in two dimensions.

Theorem 8.2 (Arbitrarily poor scaled manufacturing plans). *For $0 < \varepsilon \leq 1/4$, take*

$$A_\varepsilon = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \quad c = (1, 1)^T, \quad \ell = (L, L)^T, \quad L = \varepsilon/10, \quad b(t) = (1-t, 1-t/4)^T.$$

All resource coefficients, contributions and mandatory floors are strictly positive, and both resource capacities decrease. The unique nominal optimum is $x^0 = (q, q)$ with $q = 1/(1+\varepsilon)$. Define

$$t_* = \frac{1 - \varepsilon - (1 - \varepsilon^2)L}{1 - \varepsilon/4}, \quad B_* = 1 - t_*/4 - \varepsilon L.$$

At the feasible stress t_ , the unique optimum is (L, B_*) , whereas the nominal-surplus scaling rule has value*

$$S(t_*) = 2L + \frac{2\varepsilon(B_* - L)}{1 + \varepsilon}, \quad V(t_*) = L + B_*. \quad (21)$$

Consequently $S(t_)/V(t_*) \rightarrow 0$ and $V(t_*)/S(t_*) \rightarrow \infty$ as $\varepsilon \downarrow 0$.*

Proof. The two nominal resource equalities intersect at (q, q) , above both floors. The strictly positive dual vector $(1/(1+\varepsilon), 1/(1+\varepsilon))$ forces both equalities at every optimizer, proving uniqueness. The definition of t_* makes (L, B_*) the intersection of both stressed resource boundaries. Since $t_* < 1$ and $\varepsilon \leq 1/4$, $B_* \geq 3/4 - \varepsilon^2/10 > L$. This intersection is feasible and the same positive dual proves its optimality and uniqueness.

The nominal residual capacity in each row is $r_0 = 1 - (1+\varepsilon)L$. Resource 1 is the smaller stressed residual, so $\alpha(t_*) = (1 - t_* - (1+\varepsilon)L)/r_0 = \varepsilon(B_* - L)/r_0$. Since $q - L = r_0/(1+\varepsilon)$, each scaled output is $L + \varepsilon(B_* - L)/(1+\varepsilon)$, establishing (21). Moreover $r_0 - t_* = \varepsilon(B_* - L) > 0$, so this stress is strictly inside the feasible interval. As $\varepsilon \rightarrow 0$, $t_* \rightarrow 1$, $B_* \rightarrow 3/4$, $V(t_*) \rightarrow 3/4$ and $S(t_*) \rightarrow 0$. $\square$

The poor plan is feasible and based on the unique nominal optimum; its failure cannot be attributed to choosing a bad nominal solution or dropping the commitments. The model exposes a scarce input that is mainly consumed by one product. Uniform scaling sacrifices the other product even when reoptimization would retain it. This is a deterministic stress-path result, not a statistical assertion about how often such configurations occur.

9. Certificate-driven tracing without vertex enumeration

Explicit enumeration of $\mathcal{V}$ becomes expensive. Instead, an LP oracle at t returns $V(t)$ and a verified feasible dual y_t . The affine function

$$f_t(s) = c^T(\ell^0 + se) + y_t^T(r^0 - sh)$$

is a global upper bound and touches V at t . For sample endpoints $a < b$, define

$$U_{ab}(s) = \min\{f_a(s), f_b(s)\}, \quad L_{ab}(s) = \frac{b-s}{b-a}V(a) + \frac{s-a}{b-a}V(b). \quad (22)$$

Concavity and weak duality imply $L_{ab} \leq V \leq U_{ab}$ throughout the interval. The maximum of $U_{ab} - L_{ab}$ occurs at an endpoint or at the intersection of the two lines, when that intersection lies inside the interval.

The tracing algorithm starts with 0 and T . On an interval whose maximum gap exceeds η , it queries the line intersection and splits the interval; otherwise it retains the chord. The resulting piecewise-affine lower curve $\tilde{V}$ satisfies

$$0 \leq V(t) - \tilde{V}(t) \leq \eta \quad \forall t \in [0, T]. \quad (23)$$

In exact arithmetic and with extreme-point oracle certificates, every positive-gap query introduces a previously absent active affine branch. Finitely many dual vertices imply termination for a positive tolerance. Exact reconstruction is available when all terminal gaps vanish; the implementation instead uses a numerical tolerance and reports all residuals. The intersection-query principle has classical parametric-optimization precedents [2]; it is not presented as a new generic oracle method.

Corollary 9.1 (An interval-wide regret certificate). *Let $\tilde{R}$ be the maximum of $\tilde{V} - S$ over endpoints and the union of tracing knots and α breakpoints. If (23) holds, then*

$$\tilde{R} \leq \max_{[0, T]} (V - S) \leq \tilde{R} + \eta. \quad (24)$$

Proof. The finite-partition argument applies exactly to $\tilde{V} - S$. Subtract S from (23) and take maxima. $\square$

This certificate audits the entire interval. A dense grid of optimizer runs gives sampled losses and may miss an interior maximum; it is used for independent comparison, not as a proof of completeness. Primal and dual objective tolerances must be added to the theoretical η when a floating-point oracle is used. The supplied implementation checks dual nonnegativity, reduced costs, primal residuals and objective agreement, and identifies its numerical bracket separately from the exact mathematical result.

10. Expanded computational design and evidence

The supplementary experiment uses 90 seeded positive-resource manufacturing instances: 30 each with $(m, n) = (3, 4), (4, 6), (4, 8)$. Contributions, nominal commitments, positive resource coefficients and surplus capacities are generated independently within the stated ranges in the script. A nominal LP determines z^0 . Both capacity losses and commitment increases are nonzero, and the analysis horizon is the exact ray-feasibility limit (16). No real-factory measurements are claimed.

The tracing algorithm is compared with 101 independent optimizer calls on a uniform grid. It also enumerates candidate dual vertices by active-constraint combinations for these modest dimensions, verifies every retained vertex and evaluates their full lower envelope. This independent finite-envelope comparison checks the oracle-traced value between sample points. For each instance, all scaling breakpoints are included in the regret audit; candidate worst-stress values are evaluated again by the direct LP. The records include tracing calls, certified terminal gaps, normalized worst regret, the maximizing stress and the shortfall of the grid estimate.

Runtime and call-count comparisons are restricted to this local implementation. They do not establish polynomial-time complexity or superiority over specialized parametric simplex software. The larger study tests general dimensions and joint stress; the two-product example remains the transparent analytic unit test and exact counterexample to endpoint-only regret assessment.

Table 2: Expanded synthetic study: measured numerical summaries.

Metric	Result
Manufacturing instances	90
Mean tracing oracle calls	3.57778
Maximum tracing oracle calls	9
Instances with interior regret maximum	62
Largest 101-point grid underestimate	0.442479
Maximum LP/enumerated-envelope discrepancy	4.83169e-13

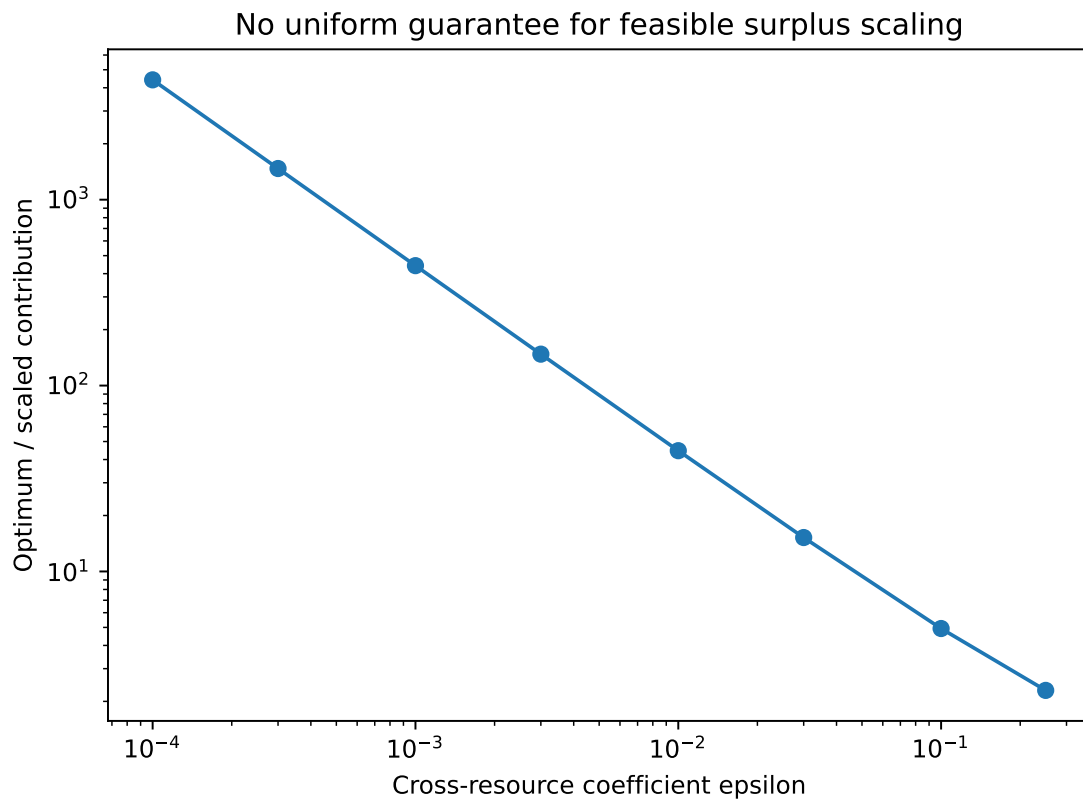


Figure 5: The optimum-to-scaled-value ratio diverges as cross-resource consumption decreases, despite positive mandatory floors.

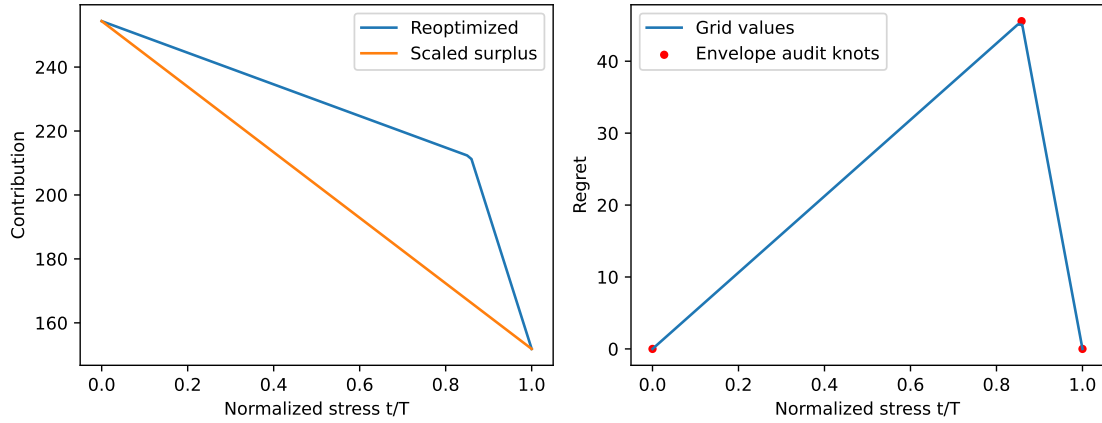


Figure 6: A higher-dimensional manufacturing example: reoptimized and scaled values, with the finite regret audit partition.

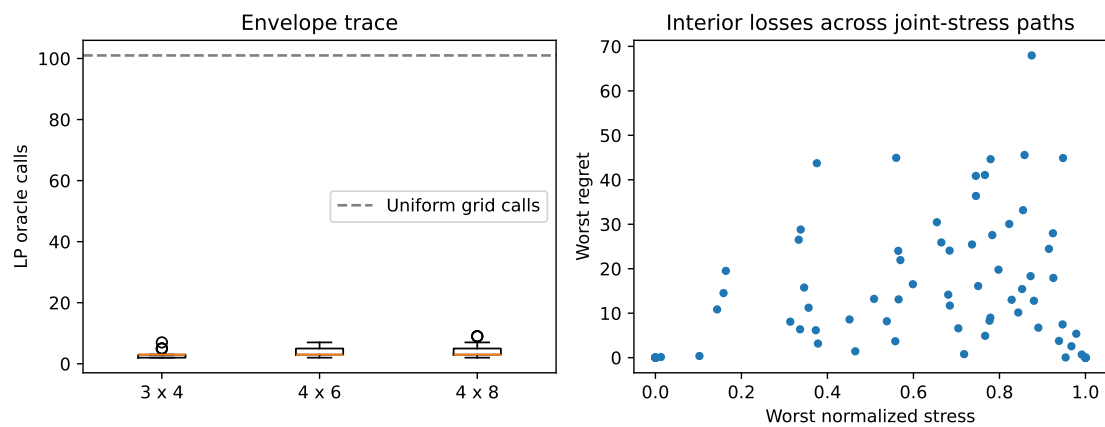


Figure 7: Oracle counts and locations of worst regret across 90 joint-stress manufacturing paths.

11. Positioning and limitations

Parametric LP already describes finite active regions [11], its worst-case complexity can be high [17], and intersection-query envelope methods have established precedents [2]. The present contribution is the commitment-adjusted scaling policy, its complete worst-regret partition, and the positive-floor family showing unbounded relative loss. The oracle procedure implements established geometry for that audit; it is not claimed as a new generic parametric algorithm.

The 90-instance study supports correctness within the stated continuous models. It does not establish industrial savings, polynomial complexity, or worldwide priority for the application-specific combination. A broader specialist literature comparison could identify additional closely related postoptimality or adjustment-policy results.

The model has divisible products, fixed contributions and no demand ceilings or inventory penalties. Whole-batch restrictions require an integer formulation. The uncertainty intervals are stipulated rather than calibrated. The mathematical results are exact under their assumptions; the computational brackets include a stated numerical allowance and are not formally verified rational proofs.

12. Conclusion

The manufacturing audit separates feasibility from economic quality along a complete joint-stress path. Uniformly scaling a unique nominal optimum can lose nearly all attainable contribution even with strictly positive coefficients and commitments. The finite regret partition and the interval-wide certificate detect losses that endpoint or grid checks can miss. Exact two-product formulas and independent higher-dimensional dual-envelope comparisons support the general result within the stated continuous model.

A. Reproduction

The package contains complete LaTeX, compiled PDF, vector and raster modelling graphics, CSV outputs, numerical summaries and an independent experiment script. Run `bash reproduce.sh`. The named author should review the proof and literature comparison before submission. All reported quantities are synthetic manufacturing inputs.

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