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A Unified Recurrence Framework for Power-Series and Frobenius Solutions of Second-Order Linear ODEs

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Abstract

This paper presents a unified coefficient-recurrence framework for obtaining series solutions of second-order linear ordinary differential equations with variable coefficients. The standard power-series method, applicable at ordinary points, and the Frobenius method, applicable at regular singular points, are treated as special cases of a common series ansatz $y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$*, where* $r = 0$ *recovers the ordinary case and* r *is determined by an indicial equation in the singular case. General recurrence relations for analytic coefficient functions are derived, conditions under which series terminate to yield polynomial solutions are established, and resonance in the integer-difference root case is characterized. Carefully selected examples illustrate the framework, including polynomial solutions, fractional exponents, resonance with logarithmic terms, and the Bessel equation. The analysis provides a systematic structural comparison of the two methods in terms of recurrence order, termination, resonance and computational complexity.*

Keywords: Power series method; Frobenius method; recurrence relations; indicial equation; resonance; Bessel equation.

MSC: 34A05, 34A25, 34A30, 34B30

1. Introduction

Consider the second-order linear ordinary differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

where $a_2(x)$, $a_1(x)$, and $a_0(x)$ are functions of the independent variable x. When these coefficients are constants, the equation admits solutions expressible as combinations of exponentials, sines, and cosines. When the coefficients are functions of x, such elementary solutions are generally unavailable. Equations of this type include the classical equations of Legendre,

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,$$

Bessel,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

and Hermite,

$$y'' - 2xy' + 2ny = 0.$$

These equations arise naturally in the solution of partial differential equations in physics and engineering, particularly in problems involving spherical, cylindrical, and parabolic coordinates.

One systematic approach to solving variable-coefficient ODEs is to seek solutions in the form of infinite power series. The standard power series method assumes a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where x_0 is the point of expansion. This method succeeds when x_0 is an ordinary point, meaning the coefficient functions $P(x)$ and $Q(x)$ in the normalized equation

$$y'' + P(x)y' + Q(x)y = 0$$

are analytic at x_0. Under these conditions, the series converges in some neighborhood of x_0 and yields two linearly independent solutions.

When x_0 is a singular point, the ordinary Taylor-series ansatz generally does not provide a complete fundamental system. A more general approach, due to Ferdinand Georg Frobenius, is then required. The Frobenius method seeks a solution of the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where r is a constant to be determined. Substitution of this series into the differential equation leads to the indicial equation, a quadratic in r of the form

$$r(r - 1) + p_0 r + q_0 = 0,$$

where p_0 and q_0 are constants determined by the coefficients of the equation. The roots of this equation determine the structure of the solution and the presence of logarithmic terms.

Although the ordinary power-series and Frobenius methods are individually well established, their coefficient recurrences are commonly presented as separate computational procedures. This separation obscures the structural relationship between the two methods, particularly the role of the leading exponent, recurrence shifts, termination conditions, and resonance.

Moreover, existing comparative treatments often emphasize procedural examples rather than a unified recurrence formulation. Classical textbooks provide thorough treatments of both methods [1, 2]. More recently, Maheshwari and Mangal [3] presented a comparative analysis of various series-solution methods for second-order ODEs with variable coefficients, including a modification to the Frobenius method that addresses certain computational difficulties. Oxford Science Trove [4] emphasizes the power series method as foundational for the study of Legendre polynomials and other special functions. The LibreTexts mathematics library offers accessible treatments of singular points and the classification of regular and irregular singularities [5].

The present work addresses the gap between procedural exposition and structural analysis. A unified coefficient-recursion framework is formulated that treats ordinary and regular singular expansion points within a common series representation. The ordinary power-series method emerges as the zero-exponent case of the generalized Frobenius expansion. At a regular singular point, the admissible leading exponents are precisely the roots of the associated indicial polynomial. Recurrence relations for analytic coefficient functions are established, resonance in the integer-difference case is characterized, and it is shown how recurrence termination, coefficient singularity, and logarithmic terms determine the resulting solution structure.

The contributions of this paper are as follows. A unified series representation for second-order linear ODEs with variable coefficients is developed, encompassing both ordinary and regular singular points. General recurrence relations for analytic and polynomial coefficient functions are derived. A complete classification of the three Frobenius cases is provided, including a rigorous treatment of resonance and logarithmic solutions. Termination conditions for series solutions are established and conditions under which polynomial solutions arise are characterized. A set of carefully selected examples demonstrates the framework in practice, including polynomial solutions, fractional exponents, resonance, and the Bessel equation. A computational algorithm for automatically generating recurrences is presented, along with a quantitative comparison of recurrence structures.

The paper is organized as follows. Section 2 presents the necessary preliminaries, including the classification of ordinary and singular points, the Wronskian and Abel's identity, and the unified series representation. Section 3 derives the general recurrence relations for both the power series and Frobenius methods, with special attention to the polynomial coefficient case. Section 4 develops the theory of the indicial equation and provides a complete analysis of the three cases arising from its roots, including the resonance phenomenon. Section 5 addresses convergence and termination conditions for series solutions. Section 6 presents a computational algorithm for generating series coefficients, including resonance detection. Section 7 provides six illustrative examples that demonstrate the theoretical framework across the full spectrum of cases. Section 8 presents a quantitative comparative analysis of the two methods. Section 9 concludes the paper and discusses directions for future work.

2. Preliminaries

2.1. Definitions

A second-order linear ODE in standard form is

$$y'' + P(x)y' + Q(x)y = R(x), \tag{2.1}$$

obtained from $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ by setting

$$P(x) = \frac{a_1(x)}{a_2(x)}, \quad Q(x) = \frac{a_0(x)}{a_2(x)}, \quad R(x) = \frac{g(x)}{a_2(x)}.$$

Equation (2.1) is *homogeneous* if $R(x) = 0$.

A power series centered at x_0 is $\sum_{n=0}^{\infty} a_n(x - x_0)^n$. A function is *analytic* at x_0 if its Taylor series converges in a neighborhood of x_0.

2.2. Ordinary and Singular Points

Definition 2.1. A point x_0 is an *ordinary point* of (2.1) if $P(x)$ and $Q(x)$ are analytic at x_0. Otherwise, it is a *singular point*. A singular point is *regular* if both

$$(x - x_0)P(x) \quad \text{and} \quad (x - x_0)^2 Q(x)$$

are analytic at x_0; otherwise, it is *irregular*.

Remark 2.2. For $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, the zeros of $a_2(x)$ identify possible singular points.

Example 2.3. For $(1 - x^2)y'' - xy' + 4y = 0$, the normalized equation is

$$y'' - \frac{x}{1 - x^2} y' + \frac{4}{1 - x^2} y = 0.$$

Thus $x = 0$ is ordinary, while $x = \pm 1$ are singular.

Example 2.4. For Bessel's equation $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$, we have $xP(x) = 1$ and $x^2 Q(x) = x^2 - \nu^2$, both analytic at $x = 0$. Hence $x = 0$ is a regular singular point.

2.3. Wronskian and Linear Independence

Definition 2.5. For solutions y_1, y_2, the Wronskian is

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

Theorem 2.6 (Abel's Identity)**.** *If y_1, y_2 solve $y'' + P(x)y' + Q(x)y = 0$, then*

$$W(y_1, y_2)(x) = C \exp\left(-\int P(x)\, dx\right).$$

In particular, y_1 and y_2 are linearly independent if $C \neq 0$.

2.4. Unified Series Representation

Both methods are unified by the ansatz

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n. \tag{2.2}$$

The power series method corresponds to $r = 0$. The Frobenius method allows arbitrary r, determined by the indicial equation. Thus the Frobenius method generalizes the power series method, and the ordinary power series is recovered as the special case $r = 0$.

2.5. Terminology

A *power series solution* has form $\sum a_n x^n$ at an ordinary point. A *Frobenius series* has form $x^r \sum a_n x^n$ at a regular singular point. A *terminating series* yields a polynomial. The *indicial equation* determines r. *Resonance* occurs when indicial roots differ by an integer, potentially producing logarithmic terms.

3. General Recurrence Relations

In this section, we derive the fundamental recurrence relations for series solutions of second-order linear ODEs, treating both the power series and Frobenius cases within a unified framework.

3.1. Recurrence Relations at an Ordinary Point

Let $x = 0$ be an ordinary point of

$$y'' + P(x)y' + Q(x)y = 0.$$

Then $P(x) = \sum_{n=0}^{\infty} p_n x^n$ and $Q(x) = \sum_{n=0}^{\infty} q_n x^n$. With

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n, \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n,$$

substitution yields the recurrence

$$(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n} p_k(n-k+1)a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k} = 0. \tag{3.1}$$

Thus

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} \left[\sum_{k=0}^{n} p_k(n-k+1)a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k} \right]. \tag{3.2}$$

For polynomial coefficients $P(x) = \sum_{k=0}^{m} p_k x^k$ and $Q(x) = \sum_{k=0}^{l} q_k x^k$, (3.1) becomes

$$(n+2)(n+1)a_{n+2} + \sum_{k=0}^{\min(m,n)} p_k(n-k+1)a_{n-k+1} + \sum_{k=0}^{\min(l,n)} q_k a_{n-k} = 0. \tag{3.3}$$

3.2. Frobenius Recurrence Relations

For a regular singular point at $x = 0$, write the equation as

$$x^2 y'' + x p(x) y' + q(x) y = 0, \tag{3.4}$$

where $p(x) = xP(x) = \sum_{n=0}^{\infty} p_n x^n$ and $q(x) = x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n$.

Seek

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0. \tag{3.5}$$

Substituting into (3.4) and collecting the coefficient of x^{n+r} gives

$$(n+r)(n+r-1)a_n + \sum_{m=0}^{n} p_{n-m}(m+r)a_m + \sum_{m=0}^{n} q_{n-m} a_m = 0. \tag{3.6}$$

For $n = 0$, we obtain the indicial equation:

$$r(r-1) + p_0 r + q_0 = 0. \tag{3.7}$$

Equivalently,

$$r^2 + (p_0 - 1)r + q_0 = 0. \tag{3.8}$$

For $n \geq 1$, (3.6) yields the Frobenius recurrence:

$$a_n = -\frac{\sum_{m=0}^{n-1} \left[p_{n-m}(m+r) + q_{n-m}\right] a_m}{(n+r)(n+r-1) + p_0(n+r) + q_0}. \tag{3.9}$$

The denominator in (3.9) factors as

$$D_n(r) = (n+r)(n+r-1) + p_0(n+r) + q_0 = (n+r-r_1)(n+r-r_2), \tag{3.10}$$

where r_1, r_2 are the indicial roots. This factorization governs resonance when $n + r = r_1$ or $n + r = r_2$.

For polynomial $p(x)$ and $q(x)$, with $p(x) = \sum_{k=0}^{m} p_k x^k$ and $q(x) = \sum_{k=0}^{l} q_k x^k$, the recurrence for $n \geq \max(m, l)$ is

$$a_n = -\frac{\sum_{k=1}^{m} p_k(n-k+r)a_{n-k} + \sum_{k=1}^{l} q_k a_{n-k}}{(n+r)(n+r-1) + p_0(n+r) + q_0}. \tag{3.11}$$

3.3. The Power Series Method as a Special Case

Setting $r = 0$ in (3.5) gives the ordinary power series $y = \sum a_n x^n$. The recurrence (3.6) becomes

$$n(n-1)a_n + \sum_{m=0}^{n} p_{n-m} m a_m + \sum_{m=0}^{n} q_{n-m} a_m = 0,$$

which, after index shifting, matches (3.1). Thus the Frobenius method contains the power series method as the special case $r = 0$.

3.4. Non-Homogeneous Equations

For

$$y'' + P(x)y' + Q(x)y = R(x), \qquad R(x) = \sum_{n=0}^{\infty} r_n x^n,$$

the recurrence (3.1) becomes

$$(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n} p_k(n-k+1)a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k} = r_n. \tag{3.12}$$

3.5. Wronskian Verification

For solutions y_1, y_2, Abel's identity gives

$$W(y_1, y_2)(x) = C \exp\left(-\int P(x)\, dx\right).$$

For two Frobenius solutions with distinct non-integer roots r_1, r_2,

$$W(y_1, y_2)(x) = C x^{r_1 + r_2 - 1} \exp\left(-\int \frac{p(x) - p_0}{x}\, dx\right),$$

with $C \neq 0$, confirming linear independence.

3.6. Summary of Recurrence Relations

The recurrences derived above are collected in Table 1 for convenient reference. The table distinguishes between ordinary and regular singular points, analytic and polynomial coefficients, and includes the non-homogeneous case.

Table 1: Summary of Recurrence Relations

Case	Recurrence
Ordinary point, analytic	$a_{n+2} = -\frac{1}{(n+2)(n+1)}\left[\sum_{k=0}^{n} p_k(n-k+1)a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k}\right]$
Ordinary point, polynomial	$(n+2)(n+1)a_{n+2} + \sum_{k=0}^{\min(m,n)} p_k(n-k+1)a_{n-k+1} + \sum_{k=0}^{\min(l,n)} q_k a_{n-k} = 0$
Regular singular, general	$a_n = -\frac{\sum_{m=0}^{n-1}[p_{n-m}(m+r)+q_{n-m}]a_m}{(n+r)(n+r-1)+p_0(n+r)+q_0}$
Regular singular, polynomial	$a_n = -\frac{\sum_{k=1}^{m} p_k(n-k+r)a_{n-k} + \sum_{k=1}^{l} q_k a_{n-k}}{(n+r)(n+r-1)+p_0(n+r)+q_0}$
Non-homogeneous	$(n+2)(n+1)a_{n+2}+\sum_{k=0}^{n} p_k(n-k+1)a_{n-k+1}+\sum_{k=0}^{n} q_k a_{n-k} = r_n$

4. Indicial Equation and Resonance

The indicial equation determines the admissible leading exponents r in Frobenius expansions. Its roots govern the structure of the solutions and the presence of logarithmic terms.

4.1. Derivation of the Indicial Equation

For the regular singular equation

$$x^2y'' + xp(x)y' + q(x)y = 0, \qquad p(x) = \sum_{n=0}^{\infty} p_n x^n, \; q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

seek $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$. Substitution gives, at $n = 0$,

$$[r(r-1) + p_0 r + q_0]\, a_0 = 0.$$

Thus

$$r^2 + (p_0 - 1)r + q_0 = 0. \tag{4.1}$$

This is the *indicial equation*. Its roots r_1, r_2 determine the solution structure.

For $n \geq 1$, the coefficient recurrence is

$$a_n = -\frac{\sum_{m=0}^{n-1} [p_{n-m}(m+r) + q_{n-m}]\, a_m}{(n+r)(n+r-1) + p_0(n+r) + q_0}. \tag{4.2}$$

The denominator factors as

$$D_n(r) = (n + r - r_1)(n + r - r_2). \tag{4.3}$$

4.2. Case 1: Distinct Roots Not Differing by an Integer

If

$$r_1 - r_2 \notin \mathbb{Z}, \tag{4.4}$$

then $D_n(r) \neq 0$ for all $n \geq 1$ at both $r = r_1$ and $r = r_2$. Thus two independent Frobenius series exist:

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n, \qquad y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n. \tag{4.5}$$

4.3. Case 2: Equal Roots

If

$$r_1 = r_2 = r, \tag{4.6}$$

the first solution is $y_1 = x^r \sum a_n x^n$. The second solution contains a logarithmic term:

$$y_2 = y_1 \ln x + x^r \sum_{n=0}^{\infty} b_n x^n. \tag{4.7}$$

This follows by differentiating the recurrence with respect to r.

4.4. Case 3: Distinct Roots Differing by an Integer

Let

$$r_1 - r_2 = N \in \mathbb{Z}^+, \qquad r_1 > r_2. \tag{4.8}$$

The larger root r_1 always yields a Frobenius series:

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n. \tag{4.9}$$

For $r = r_2$, the denominator in (4.2) has factor $n - N$, vanishing at $n = N$. The recurrence at $n = N$ gives

$$0 \cdot a_N = \sum_{m=0}^{N-1} \left[p_{N-m}(m + r_2) + q_{N-m} \right] a_m. \tag{4.10}$$

Definition 4.1 (Resonance Condition)**.** The smaller-root Frobenius series exists without a logarithmic term if and only if

$$\sum_{m=0}^{N-1} \left[p_{N-m}(m + r_2) + q_{N-m} \right] a_m = 0. \tag{4.11}$$

If (4.11) fails, the second solution requires a logarithmic term:

$$y_2 = C y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n, \qquad C \neq 0. \tag{4.12}$$

Example 4.2 (Resonance in $xy'' + 3y' - y = 0$)**.** Here $p(x) = 3$, $q(x) = -x$, so $p_0 = 3$, $q_0 = 0$, $q_1 = -1$. The indicial equation is

$$r^2 + 2r = r(r + 2) = 0,$$

so $r_1 = 0$, $r_2 = -2$, $N = 2$.

For $r = r_2 = -2$, the recurrence at $n = 1$ gives $a_1 = -a_0$. At $n = 2$, the resonance sum is

$$\sum_{m=0}^{1} [p_{2-m}(m-2) + q_{2-m}]a_m = -a_1.$$

This equals $a_0 \neq 0$, so (4.11) fails. Hence the second solution contains $\ln x$.

4.5. Reduction of Order in the Resonant Case

When (4.11) fails, construct y_2 via reduction of order. For $y'' + P(x)y' + Q(x)y = 0$,

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx. \tag{4.13}$$

With $P(x) = p(x)/x$, this gives

$$y_2 = y_1 \int x^{-p_0-2r_1} \sum_{n=0}^{\infty} d_n x^n\,dx,$$

which yields a logarithmic term when the exponent is -1.

4.6. Summary of the Three Cases

Table 2: Classification of Frobenius Solutions

Case	Condition	Solution Form
1: Distinct, non-integer	$r_1 - r_2 \notin \mathbb{Z}$	$y_1 = x^{r_1}\sum a_n x^n$, $y_2 = x^{r_2}\sum b_n x^n$
2: Equal roots	$r_1 = r_2 = r$	$y_1 = x^r\sum a_n x^n$, $y_2 = y_1 \ln x + x^r \sum b_n x^n$
3: Integer difference	$r_1 - r_2 = N \in \mathbb{Z}^+$	If (4.11) holds: $y_1 = x^{r_1}\sum a_n x^n$, $y_2 = x^{r_2}\sum b_n x^n$ If (4.11) fails: $y_1 = x^{r_1}\sum a_n x^n$, $y_2 = Cy_1 \ln x + x^{r_2}\sum b_n x^n$

4.7. Wronskian Verification

For two Frobenius solutions, Abel's identity gives

$$W(y_1, y_2)(x) = Cx^{-p_0}e^{-\int \frac{p(x)-p_0}{x}\,dx},$$

which is non-zero for $x \neq 0$, confirming linear independence. In the resonant case, the logarithmic term ensures non-zero Wronskian.

5. Convergence and Termination

We address two essential questions: the radius of convergence of series solutions and the conditions under which series terminate to yield polynomials.

5.1. Radius of Convergence

For the equation $y'' + P(x)y' + Q(x)y = 0$ with $x = 0$ an ordinary point, if $P(x)$ and $Q(x)$ are analytic in $|x| < R$, then any power series solution $\sum a_n x^n$ converges for $|x| < R$. The radius is determined by the nearest singularity of P or Q in the complex plane.

For a regular singular point, the Frobenius series $y = x^r \sum a_n x^n$ converges in a punctured neighborhood $0 < |x| < R$, where R is the radius of convergence of $p(x) = xP(x)$ and $q(x) = x^2 Q(x)$.

5.2. Termination Conditions

A series terminates if $a_n = 0$ for all $n \geq N$, yielding a polynomial solution. The Mann iteration, as analyzed in [6] for Volterra operators, provides a complementary perspective on convergence and termination in iterative fixed-point methods for nonlinear equations.

Theorem 5.1 (Termination Condition). *If the recurrence has the form $a_{n+k} = R(n)a_n$ with $R(n)$ rational, and $R(N) = 0$ for some $N \in \mathbb{N}$, then the series terminates at $n = N$.*

Example 5.2. For Legendre's equation

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0,$$

the recurrence is

$$a_{n+2} = -\frac{(\ell - n)(\ell + n + 1)}{(n + 1)(n + 2)} a_n.$$

For $\ell \in \mathbb{Z}^+$, $R(\ell) = 0$, so the series terminates, yielding the Legendre polynomials $P_\ell(x)$.

In the Frobenius method, termination occurs when the numerator in (4.2) vanishes for some n with non-zero denominator. This is precisely the resonance condition from Section 4.

5.3. Polynomial Solutions

Terminating series yield the classical orthogonal polynomials: Legendre $P_n(x)$, Hermite $H_n(x)$, Laguerre $L_n(x)$, and Chebyshev $T_n(x)$. These arise from equations with polynomial coefficients and play a central role in mathematical physics. For a complementary treatment of polynomial solutions and terminating series in the context of nonlinear integral equations in L^p spaces, see the generalized hybrid fixed point framework of Edem et al. [7].

5.4. Convergence Comparison

Table 3: Convergence Comparison of Power Series and Frobenius Methods

Method	**Point Type**	**Convergence**
Power Series	Ordinary	Converges up to nearest singularity of P or Q.
Frobenius	Regular singular	Converges in $0 < \|x\| < R$, determined by p and q.
Power Series	Regular singular	Generally fails to provide complete fundamental system.

6. Computational Algorithm

The recurrence relations derived in Sections 3 and 4 provide a systematic framework for generating series solutions of second-order linear ODEs. In this section, we present an algorithm that automates the generation of coefficients for both the power series and Frobenius methods, including resonance detection.

6.1. Algorithm Overview

The algorithm takes as input the coefficient functions of the differential equation and the desired expansion point, and returns the series solution coefficients up to a specified order. The procedure consists of four main stages:

1. Classification of the expansion point (ordinary or singular)
2. If singular, classification as regular or irregular
3. For regular singular points, computation of the indicial roots
4. Generation of coefficients via the appropriate recurrence

6.2. Pseudocode for the Power Series Method

Algorithm 1 Power Series Coefficient Generation

Require: $P(x), Q(x)$ analytic at $x = 0$; order N
Ensure: Coefficients $a_0, a_1, \ldots, a_N$

Set $a_0 = 1$, $a_1 = 0$ for first solution
Set $a_0 = 0$, $a_1 = 1$ for second solution
for $n = 0$ **to** $N - 2$ **do**
  Compute p_k, q_k for $k = 0, 1, \ldots, n$
  $S_1 \leftarrow \sum_{k=0}^{n} p_k(n - k + 1)a_{n-k+1}$
  $S_2 \leftarrow \sum_{k=0}^{n} q_k a_{n-k}$
  $a_{n+2} \leftarrow -\frac{S_1+S_2}{(n+2)(n+1)}$
end for
return $\{a_0, a_1, \ldots, a_N\}$

6.3. Pseudocode for the Frobenius Method

Algorithm 2 Frobenius Coefficient Generation

Require: $p(x) = xP(x)$, $q(x) = x^2Q(x)$ analytic at $x = 0$; order N
Ensure: Coefficients $a_0, a_1, \ldots, a_N$

Compute $p_0, p_1, \ldots, p_N$ and $q_0, q_1, \ldots, q_N$
Solve indicial equation: $r^2 + (p_0 - 1)r + q_0 = 0$
Let roots be r_1, r_2
Set $a_0 = 1$
for r **in** $\{r_1, r_2\}$ **do**
    if $r_1 - r_2 \in \mathbb{Z}$ and $r = r_2$ **then**
        Check resonance condition at $n = r_1 - r_2$
        if resonance condition fails **then**
            continue to next root (logarithm required)
        end if
    end if
    for $n = 1$ **to** N **do**
        $S \leftarrow \sum_{m=0}^{n-1} [p_{n-m}(m + r) + q_{n-m}]a_m$
        $D \leftarrow (n + r)(n + r - 1) + p_0(n + r) + q_0$
        if $D = 0$ **then**
            flag resonance at n
            break
        else
            $a_n \leftarrow -S/D$
        end if
    end for
end for
return coefficients for each valid root

6.4. Resonance Detection

A critical feature of the Frobenius algorithm is resonance detection. When the indicial roots differ by an integer, the denominator in the recurrence may vanish at a specific index. This occurs when $r_1 - r_2 = N \in \mathbb{Z}^+$, causing the recurrence for the smaller root r_2 to break down at $n = N$. The obstruction is resolved only if the resonance condition (26) holds; otherwise, the second solution necessarily contains a logarithmic term. The following algorithm automates this detection process, determining whether a second Frobenius series exists or whether reduction of order is required [8].

Algorithm 3 Resonance Detection

Require: Indicial roots r_1, r_2 with $r_1 > r_2$, coefficients p_n, q_n
Ensure: Resonance status

```
N ← r_1 − r_2
if N ∉ ℤ^+ then
    return no resonance
end if
Compute a_0, a_1, ..., a_{N−1} for r = r_2
Compute resonance sum:
R ← Σ_{m=0}^{N−1} [p_{N−m}(m + r_2) + q_{N−m}] a_m
if R = 0 then
    return resonance but series exists
else
    return resonance requiring logarithm
end if
```

6.5. Computational Complexity

Let $C(N)$ denote the number of arithmetic operations required to generate N coefficients.

For the power series method, the recurrence involves sums over n terms. The total complexity is

$$C_{\text{power}}(N) = \sum_{n=0}^{N} O(n) = O(N^2).$$

For polynomial coefficients of degree m, the sums are bounded, giving

$$C_{\text{power, poly}}(N) = O(mN).$$

For the Frobenius method, the complexity is similar, with an additional $O(N)$ cost for resonance detection:

$$C_{\text{Frobenius}}(N) = O(N^2) + O(N) = O(N^2).$$

Table 4: Computational Complexity Comparison

Method	Coefficient Type	Complexity
Power Series	General analytic	$O(N^2)$
Power Series	Polynomial, degree m	$O(mN)$
Frobenius	General analytic	$O(N^2)$
Frobenius	Polynomial, degree m	$O(mN)$

6.6. Implementation Notes

The algorithm can be implemented in any computational mathematics environment. Several practical considerations apply:

1. The coefficients p_n and q_n are typically computed from the Taylor expansions of $p(x)$ and $q(x)$. For polynomial coefficients, these are known exactly.

2. For high-order coefficient generation, arbitrary precision arithmetic may be required to avoid numerical instability.

3. The recurrence denominators $(n+r)(n+r-1)+p_0(n+r)+q_0$ may be very small for large n, requiring careful numerical handling.

4. In the resonant case, the logarithmic term can be constructed using reduction of order, as described in Subsection 4.5.

For related computational frameworks implementing iterative methods for nonlinear equations, see the adaptive Mann solver of Edem et al. [9], which combines Anderson acceleration with dynamic stopping criteria for Volterra integral equations, demonstrating the practical viability of pseudo-contractive fixed-point iterations in high-performance computing environments.

6.7. Extension to Non-Homogeneous Equations

For non-homogeneous equations, the algorithm is modified by including the forcing term. If

$$y'' + P(x)y' + Q(x)y = R(x),$$

with $R(x) = \sum_{n=0}^{\infty} r_n x^n$, the recurrence becomes

$$a_{n+2} = \frac{r_n - \sum_{k=0}^{n} p_k(n-k+1)a_{n-k+1} - \sum_{k=0}^{n} q_k a_{n-k}}{(n+2)(n+1)}.$$

The Frobenius case is more delicate, as the forcing term may introduce additional terms in the expansion. The general approach is to seek a particular solution of the form

$$y_p(x) = x^s \sum_{n=0}^{\infty} b_n x^n,$$

where s is determined by the leading behavior of $R(x)$.

Remark 6.1. The algorithms presented in this section are intended for theoretical analysis and pedagogical clarity. Production implementations would incorporate additional features such as automatic differentiation, root finding, and error estimation.

7. Illustrative Examples

In this section, we present a carefully selected set of examples that illustrate the theoretical framework developed in the preceding sections. The examples are chosen to demonstrate the full range of possibilities: ordinary points with infinite and terminating series, non-homogeneous equations, regular singular points with distinct non-integer roots, resonance with logarithmic terms, and connection to special functions.

7.1. Example A: Ordinary Point with Infinite Series

Consider

$$y'' + x^2 y = 0. \tag{7.1}$$

Here $P(x) = 0$ and $Q(x) = x^2$, both analytic everywhere, so $x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Shifting indices and equating coefficients:

$$(n+2)(n+1)a_{n+2} + a_{n-2} = 0, \qquad n \geq 2.$$

Thus

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)}, \qquad n \geq 2. \tag{7.2}$$

This is a four-step recurrence. The even and odd solutions separate:

$$a_{4k} = \frac{(-1)^k a_0}{\prod_{j=1}^{k}(4j)(4j-1)}, \qquad a_{4k+1} = \frac{(-1)^k a_1}{\prod_{j=1}^{k}(4j)(4j+1)}.$$

The general solution is

$$y = a_0\left(1 - \frac{x^4}{12} + \frac{x^8}{12 \cdot 8 \cdot 7} - \cdots\right) + a_1\left(x - \frac{x^5}{20} + \frac{x^9}{20 \cdot 9 \cdot 8} - \cdots\right).$$

The series converge for all x, as $P(x)$ and $Q(x)$ have no singularities.

7.2. Example B: Terminating Series and Polynomial Solutions

Consider

$$(1-x^2)y'' - xy' + 4y = 0. \tag{7.3}$$

In standard form,

$$y'' - \frac{x}{1-x^2}y' + \frac{4}{1-x^2}y = 0.$$

At $x = 0$, both coefficient functions are analytic, so $x = 0$ is an ordinary point. However, the normalized coefficients have singularities at $x = \pm 1$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution yields

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0.$$

Shifting indices:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}[n(n-1) + n - 4]a_n x^n = 0.$$

Since $n(n-1) + n - 4 = n^2 - 4 = (n-2)(n+2)$, we have

$$(n+2)(n+1)a_{n+2} - (n-2)(n+2)a_n = 0, \qquad n \geq 2.$$

Thus

$$a_{n+2} = \frac{n-2}{n+1}a_n, \qquad n \geq 2. \tag{7.4}$$

For $n = 0$:

$$2a_2 + 4a_0 = 0 \Rightarrow a_2 = -2a_0.$$

For $n = 1$:

$$6a_3 + 3a_1 = 0 \Rightarrow a_3 = -\frac{1}{2}a_1.$$

For $n = 2$: $a_4 = 0$. For $n = 3$: $a_5 = \frac{1}{4}a_3 = -\frac{1}{8}a_1$. For $n \geq 4$: all coefficients vanish. Therefore,

$$y = a_0(1 - 2x^2) + a_1\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5\right).$$

This solution terminates, yielding polynomial solutions. The normalized coefficient singularities at $x = \pm 1$ do not affect the solutions because the series terminates before reaching these points.

7.3. Example C: Non-Homogeneous Equation

Consider

$$y'' + 4y = x. \tag{7.5}$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = x.$$

Shifting indices:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 4\sum_{n=0}^{\infty} a_n x^n = x.$$

Equating coefficients: For $n = 0$: $2a_2 + 4a_0 = 0 \Rightarrow a_2 = -2a_0$. For $n = 1$: $6a_3 + 4a_1 = 1 \Rightarrow a_3 = \frac{1-4a_1}{6}$. For $n \geq 2$:

$$a_{n+2} = -\frac{4a_n}{(n+1)(n+2)}, \qquad n \geq 2. \tag{7.6}$$

Thus

$$y = a_0\left(1 - 2x^2 + \frac{2}{3}x^4 - \cdots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \cdots\right) + \left(\frac{x^3}{6} - \frac{x^5}{30} + \cdots\right).$$

The last series is the particular solution. The homogeneous solutions are recognized as $\cos 2x$ and $\sin 2x$ from their Taylor expansions.

7.4. Example D: Frobenius Method with Distinct Non-Integer Roots

Consider

$$3xy'' + y' - y = 0. \tag{7.7}$$

In standard form,

$$y'' + \frac{1}{3x}y' - \frac{1}{3x}y = 0.$$

Thus $p(x) = xP(x) = \frac{1}{3}$ and $q(x) = x^2Q(x) = -\frac{x}{3}$. We have $p_0 = \frac{1}{3}$ and $q_0 = 0$. The indicial equation is

$$r(r-1) + \frac{1}{3}r = 0 \Rightarrow r^2 - \frac{2}{3}r = 0 \Rightarrow r\left(r - \frac{2}{3}\right) = 0.$$

Thus

$$r_1 = \frac{2}{3}, \qquad r_2 = 0, \qquad r_1 - r_2 = \frac{2}{3} \notin \mathbb{Z}.$$

Let $y = \sum_{n=0}^{\infty} C_n x^{n+r}$. The recurrence from Section 3 is

$$C_n = \frac{C_{n-1}}{(n+r)(3n+3r-2)}, \qquad n \geq 1. \tag{7.8}$$

For $r_1 = \frac{2}{3}$:

$$C_n = \frac{C_{n-1}}{(n+\frac{2}{3})(3n+2)} = \frac{C_{n-1}}{n(3n+2)}.$$

Thus

$$C_1 = \frac{C_0}{5}, \qquad C_2 = \frac{C_0}{5 \cdot 8 \cdot 2} = \frac{C_0}{80}, \qquad C_n = \frac{C_0}{n! \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)}.$$

So

$$y_1 = C_0 x^{2/3} \left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{80 \cdot 11 \cdot 3} + \cdots \right).$$

For $r_2 = 0$:

$$C_n = \frac{C_{n-1}}{n(3n-2)}.$$

Thus

$$C_1 = C_0, \qquad C_2 = \frac{C_0}{8}, \qquad C_n = \frac{C_0}{n! \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}.$$

So

$$y_2 = C_0 \left(1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \cdots \right).$$

The general solution is $y = c_1 y_1(x) + c_2 y_2(x)$.

7.5. Example E: Resonance and Logarithmic Solutions

Consider

$$xy'' + 3y' - y = 0. \tag{7.9}$$

In standard form,

$$y'' + \frac{3}{x} y' - \frac{1}{x} y = 0.$$

Thus $p(x) = 3$ and $q(x) = -x$. We have $p_0 = 3$, $q_0 = 0$, $q_1 = -1$. The indicial equation is

$$r(r-1) + 3r = r^2 + 2r = r(r+2) = 0.$$

Thus

$$r_1 = 0, \qquad r_2 = -2, \qquad r_1 - r_2 = 2 \in \mathbb{Z}^+.$$

For $r_1 = 0$, the recurrence is

$$C_n = \frac{C_{n-1}}{n(n+2)}, \qquad n \geq 1.$$

Thus

$$C_1 = \frac{C_0}{3}, \qquad C_2 = \frac{C_0}{24}, \qquad C_n = \frac{2C_0}{n!(n+2)!}.$$

So

$$y_1 = C_0\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \cdots\right). \tag{7.10}$$

For $r_2 = -2$, the recurrence is

$$C_n = \frac{C_{n-1}}{(n-2)n}, \qquad n \geq 1.$$

For $n = 1$:

$$C_1 = \frac{C_0}{(-1)(1)} = -C_0.$$

For $n = 2$, the denominator vanishes:

$$0 \cdot C_2 = C_1 = -C_0.$$

Since $C_0 \neq 0$, this is impossible. The resonance condition fails, and a logarithmic term is required.

Using reduction of order with $P(x) = \frac{3}{x}$:

$$y_2 = y_1(x)\int \frac{e^{-\int \frac{3}{x}\,dx}}{y_1^2(x)}\,dx = y_1(x)\int \frac{x^{-3}}{y_1^2(x)}\,dx.$$

With y_1 from (7.10),

$$y_1^2 = 1 + \frac{2}{3}x + \frac{7}{36}x^2 + \frac{1}{30}x^3 + \cdots,$$

and

$$\frac{1}{y_1^2} = 1 - \frac{2}{3}x + \frac{1}{4}x^2 - \frac{19}{270}x^3 + \cdots.$$

Thus

$$y_2 = y_1(x)\left(-\frac{1}{2x^2} + \frac{2}{3x} + \frac{1}{4}\ln x - \frac{19}{270}x + \cdots\right).$$

The general solution is $y = c_1 y_1(x) + c_2 y_2(x)$.

7.6. Example F: Bessel's Equation

Consider Bessel's equation of order ν:

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \tag{7.11}$$

Here $p(x) = 1$ and $q(x) = x^2 - \nu^2$. Thus $p_0 = 1$, $q_0 = -\nu^2$, and $q_2 = 1$. The indicial equation is

$$r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0,$$

so

$$r_1 = \nu, \qquad r_2 = -\nu.$$

For $r = \nu$, the recurrence from Section 3 is

$$a_n = -\frac{a_{n-2}}{(n+\nu)^2 - \nu^2} = -\frac{a_{n-2}}{n(n+2\nu)}, \qquad n \geq 2.$$

The odd coefficients vanish, and the even coefficients satisfy

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(\nu+1)(\nu+2)\cdots(\nu+k)}.$$

Choosing $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$ gives

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{7.12}$$

For non-integer ν, the second solution is $J_{-\nu}(x)$. For integer $\nu = n$, the roots differ by $2n \in \mathbb{Z}$, resonance occurs, and the second solution is the Bessel function of the second kind $Y_n(x)$, which contains a logarithmic term.

7.7. Summary of Examples

The following table summarizes the key features of each example.

Table 5: Summary of Examples

Example	Equation	Point Type	Key Feature
A	$y'' + x^2 y = 0$	Ordinary	Infinite series, four-step recurrence
B	$(1-x^2)y'' - xy' + 4y = 0$	Ordinary	Terminating series, polynomial solutions
C	$y'' + 4y = x$	Ordinary	Non-homogeneous, particular solution
D	$3xy'' + y' - y = 0$	Regular singular	Distinct non-integer roots
E	$xy'' + 3y' - y = 0$	Regular singular	Resonance, logarithmic term
F	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$	Regular singular	Bessel functions, special functions

Remark 7.1. These six examples collectively illustrate the full range of phenomena associated with series solutions of second-order linear ODEs with variable coefficients. The ordinary-point examples (A-C) demonstrate the power series method in both homogeneous and non-homogeneous settings. The regular singular point examples (D-F) demonstrate the Frobenius method in all three cases: distinct non-integer roots, integer-difference roots with resonance, and the connection to special functions.

8. Comparative Analysis

The preceding sections have developed a unified framework for series solutions of second-order linear ODEs with variable coefficients. We now present a quantitative comparison of the power series and Frobenius methods based on the analytical structure of the recurrences.

8.1. Recurrence Structure Comparison

For the general equation

$$y'' + P(x)y' + Q(x)y = 0,$$

with $P(x) = \sum p_n x^n$ and $Q(x) = \sum q_n x^n$, the power series recurrence at an ordinary point is

$$a_{n+2} = -\frac{1}{(n+2)(n+1)}\left[\sum_{k=0}^{n} p_k(n-k+1)a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k}\right]. \tag{8.1}$$

For a regular singular point, with $p(x) = xP(x) = \sum p_n x^n$ and $q(x) = x^2Q(x) = \sum q_n x^n$, the Frobenius recurrence is

$$a_n = -\frac{\sum_{m=0}^{n-1} \left[p_{n-m}(m+r) + q_{n-m}\right] a_m}{(n+r)(n+r-1) + p_0(n+r) + q_0}. \tag{8.2}$$

The fundamental relationship is that (8.1) is the special case of (8.2) with $r = 0$.

8.2. Quantitative Comparison

The following table provides a structured comparison across multiple dimensions. Recent algorithmic developments in Banach spaces, including inertial norm-free methods for split common fixed-point problems [10], have demonstrated the effectiveness of acceleration techniques in iterative schemes. Also, Strong convergence results for Halpern iterations in ordered Banach spaces [11] demonstrate the importance of order-theoretic and geometric conditions in establishing convergence, paralleling the structural comparison presented here.

Table 6: Quantitative Comparison of Series Methods

Criterion	**Power Series Method**	**Frobenius Method**
Applicable point	Ordinary: P, Q analytic	Regular singular: xP, x^2Q analytic
Solution ansatz	$\sum a_n x^n$	$x^r \sum a_n x^n$
Leading exponent	Fixed $r = 0$	Determined by indicial equation
Free parameters	a_0, a_1 (two constants)	a_0 for each root; second root may require logarithm
Recurrence order	Two-step: $a_{n+2} = F(n)a_n$	One-step: $a_n = G(n,r)a_{n-1}$ for polynomial coefficients
Termination	Occurs when $F(N) = 0$	Occurs when resonance condition satisfied
Logarithmic terms	Never required	Required when roots equal or differ by integer with obstruction
Complexity	$O(N^2)$ general, $O(mN)$ polynomial	$O(N^2)$ general, $O(mN)$ polynomial
Special functions	Legendre, Chebyshev polynomials	Bessel, hypergeometric functions

8.3. Computational Efficiency

For polynomial coefficients, both methods reduce to recurrences of the form

$$a_{n+k} = R(n)a_n,$$

where k depends on the degree structure. The efficiency of generating N coefficients is comparable for both methods, with complexity $O(mN)$ where m is the polynomial degree.

The Frobenius method requires the additional step of solving the indicial equation and checking the resonance condition. This adds a constant overhead independent of N and is negligible for large N.

8.4. When to Use Each Method

The choice of method is determined by the nature of the expansion point:

1. If x_0 is an ordinary point, the power series method is appropriate and simpler.
2. If x_0 is a regular singular point, the Frobenius method is required.
3. If x_0 is an irregular singular point, neither method applies directly.

When the Frobenius method yields roots differing by an integer, the resonance analysis determines whether a second pure Frobenius series exists or whether a logarithmic term is required.

9. Conclusion

This paper has presented a unified coefficient-recurrence framework for series solutions of second-order linear ODEs with variable coefficients. The power series and Frobenius methods are treated as special cases of the common ansatz $y(x) = x^r \sum a_n x^n$, with $r = 0$ recovering the ordinary case.

The principal findings are as follows. The general recurrences (8.1) and (8.2) establish the structural relationship between the methods, with the power series recurrence obtained by setting $r = 0$ in the Frobenius recurrence. The indicial equation (4.1) determines the admissible leading exponents at a regular singular point, with the three root cases governing the solution structure and logarithmic terms. The resonance condition (4.11) characterizes when the smaller-root Frobenius series exists; failure requires a logarithmic term, as demonstrated in Example E. Termination of series solutions is governed by vanishing recurrence factors, yielding polynomial solutions and connecting to classical orthogonal polynomials. The computational complexity is $O(N^2)$ for general analytic coefficients and $O(mN)$ for polynomial coefficients of degree m, with the Frobenius method adding constant overhead for resonance detection.

The six examples in Section 7 illustrate the framework across the full spectrum: ordinary points, non-homogeneous equations, non-integer roots, resonance with logarithmic terms, and the Bessel equation.

A limitation of the present work is the restriction to second-order equations. Extension to higher-order equations, systems of ODEs, and PDEs with memory effects [12] would be natural directions for future research. The computational algorithm in Section 6 could also be refined with automatic differentiation and error estimation.

The unified framework established here provides both a pedagogical resource for students and a practical reference for researchers working with variable-coefficient differential equations.

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